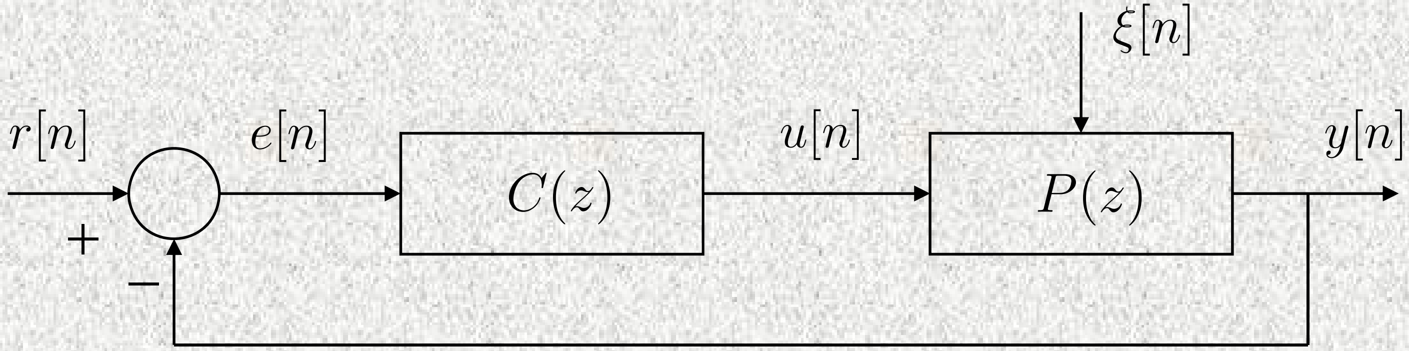
Digital PID Controller Design



- Plant and Controller

$$G(z) = \frac{N(z)}{D(z)}, \qquad C(z) = \frac{N_C(z)}{D_C(z)}.$$

- The *characteristic polynomial* of the closed loop system

$$\Pi(z) := D_C(z)D(z) + N_C(z)N(z)$$

TCHEBYSHEV REPRESENTATION AND ROOT CLUSTERING

Tchebyshev representation of real polynomials

- Consider a real polynomial $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$

- The image of $P(z)$ evaluated on the circle $\mathcal{C}_\rho$ of radius ρ, centered at the origin is:

$$\left\{ P(z) \ : \ z = \rho e^{j\theta}, \quad 0 \leq \theta \leq 2\pi \right\}.$$

- As the coefficients a_i are real $P\left(\rho e^{j\theta}\right)$ and $P\left(\rho e^{-j\theta}\right)$ are conjugate complex numbers, and so it suffices to determine the image of the upper half of the circle:

$$\left\{ P(z) \ : \ z = \rho e^{j\theta}, \quad 0 \leq \theta \leq \pi \right\}.$$

- Since $\left. z^k \right|_{z=\rho e^{j\theta}} = \rho^k(\cos k\theta + j \sin k\theta)$,

$$
\begin{aligned}
P\left(\rho e^{j\theta}\right) &= \underbrace{\left(a_n \rho^n \cos n\theta + \cdots + a_1 \rho \cos\theta + a_0\right)}_{\bar{R}(\rho,\theta)} + j \underbrace{\left(a_n \rho^n \sin n\theta + \cdots + a_1 \rho \sin\theta\right)}_{\bar{I}(\rho,\theta)} \\
&= \bar{R}(\rho,\theta) + j\bar{I}(\rho,\theta).
\end{aligned}
$$

- Consider $(\rho e^{j\theta})^k = \rho^k \cos k\theta + j\rho^k \sin k\theta$

- Write $u = -\cos\theta$ and define the generalized Tchebyshev polynomials as follows:

$$
c_k(u,\rho) = \rho^k c_k(u), \quad s_k(u,\rho) = \rho^k s_k(u), \quad k = 0, 1, 2 \cdots
$$

and note that

$$
\begin{aligned}
s_k(u,\rho) &= -\frac{1}{k} \cdot \frac{d\left[c_k(u,\rho)\right]}{du}, \qquad k = 1, 2, \cdots \\
c_{k+1}(u,\rho) &= -\rho u c_k(u,\rho) - \left(1 - u^2\right) \rho s_k(u,\rho), \qquad k = 1, 2, \cdots
\end{aligned}
$$

- The generalized Tchebyshev polynomials are for $k = 1, \cdots 5$:

k	$c_k(u,\rho)$	$s_k(u,\rho)$
1	$-\rho u$	ρ
2	$\rho^2\left(2u^2-1\right)$	$-2\rho^2 u$
3	$\rho^3\left(-4u^3+3u\right)$	$\rho^3\left(4u^2-1\right)$
4	$\rho^4\left(8u^4-8u^2+1\right)$	$\rho^4\left(-8u^3+4u\right)$
5	$\rho^5\left(-16u^5+20u^3-5u\right)$	$\rho^5\left(16u^4-12u^2+1\right)$
$\vdots$	$\vdots$	$\vdots$

- With this notation, $P\left(\rho e^{j\theta}\right) = R(u,\rho) + j\sqrt{1-u^2}T(u,\rho) =: P_c(u,\rho)$ where

$$
\begin{aligned}
R(u,\rho) &= a_n c_n(u,\rho) + a_{n-1}c_{n-1}(u,\rho) + \cdots + a_1 c_1(u,\rho) + a_0 \\
T(u,\rho) &= a_n s_n(u,\rho) + a_{n-1}s_{n-1}(u,\rho) + \cdots + a_1 s_1(u,\rho).
\end{aligned}
$$

- $R(u,\rho)$ and $T(u,\rho)$ are polynomials in u and ρ.

- The complex plane image of $P(z)$ as z traverses the upper half of the circle $\mathcal{C}_\rho$ can be obtained by evaluating $P_c(u,\rho)$ as u runs from -1 to $+1$.

LEMMA

For a fixed $\rho > 0$,

(a) if $P(z)$ has no roots on the circle of radius $\rho > 0$,
$(R(u,\rho), T(u,\rho))$ have no common roots for $u \in [-1,1]$ and $R(\pm 1,\rho) \neq 0$.

(b) if $P(z)$ has $2m$ roots at $z = -\rho\,(z = +\rho)$,
then $R(u,\rho)$ and $T(u,\rho)$ have m roots each at $u = +1$ $(u = -1)$.

(c) if $P(z)$ has $2m-1$ roots at $z = -\rho\,(z = +\rho)$, then $R(u,\rho)$ and $T(u,\rho)$ have m and $m-1$ roots, respectively at $u = +1$ $(u = -1)$.

(d) if $P(z)$ has q_i pairs of complex roots at $z = -\rho u_i \pm j\rho\sqrt{1-u_i^2}$, for $u_i \neq \pm 1$, then $R(u,\rho)$ and $T(u,\rho)$ each have q_i real roots at $u = u_i$.

- When the circle of interest is the unit circle, that is $\rho = 1$, we will write $P_c(u,1) = P_c(u)$ and also

$$R(u,1) =: R(u), \qquad T(u,1) =: T(u).$$

Interlacing Conditions for Root Clustering and Schur Stability

THEOREM

$P(z)$ has all its zeros strictly within $\mathcal{C}_\rho$ if and only if

(a) $R(u,\rho)$ has n real distinct zeros r_i, $i = 1, 2, \cdots, n$ in $(-1, 1)$.

(b) $T(u,\rho)$ has $n-1$ real distinct zeros t_j, $j = 1, 2, \cdots, n-1$ in $(-1, 1)$.

(c) The zeros r_i and t_j interlace:

$$-1 < r_1 < t_1 < r_2 < t_2 < \cdots < t_{n-1} < r_n < +1.$$

The three conditions given in the above Theorem may be referred to as interlacing conditions on $R(u,\rho)$ and $T(u,\rho)$. By setting $\rho = 1$ in the above Theorem we obtain conditions for Schur stability in terms of interlacing of the zeros of $R(u)$ and $T(u)$.

Tchebyshev Representation of Rational Functions

- Let

$$P_i(z)|_{z=-\rho u+j\rho\sqrt{1-u^2}} = R_i(u,\rho)+j\sqrt{1-u^2}T_i(u,\rho), i=1,2.$$

-

$$Q(z)|_{z=-\rho u+j\rho\sqrt{1-u^2}} = \left.\frac{P_1(z)P_2\left(z^{-1}\right)}{P_2(z)P_2\left(z^{-1}\right)}\right|_{z=-\rho u+j\rho\sqrt{1-u^2}}$$

$$= \frac{\overbrace{\left(R_1(u,\rho)R_2(u,\rho)+\left(1-u^2\right)T_1(u,\rho)T_2(u,\rho)\right)}^{R(u,\rho)}}{R_2^2(u,\rho)+\left(1-u^2\right)T_2^2(u,\rho)}$$

$$+j\frac{\sqrt{1-u^2}\overbrace{\left(T_1(u,\rho)R_2(u,\rho)-R_1(u,\rho)T_2(u,\rho)\right)}^{T(u,\rho)}}{R_2^2(u,\rho)+\left(1-u^2\right)T_2^2(h,\rho)}.$$

- $R(u,\rho)$, $T(u,\rho)$ are rational functions of the real variable u which runs from -1 to +1.

ROOT COUNTING FORMULAS

LEMMA

Let the real polynomial $P(z)$ have i roots in the interior of the circle $\mathcal{C}_\rho$ and no roots on the circle. Then:

$$\Delta_0^\pi[\phi_P(\theta)] = \pi i$$

LEMMA

Let $Q(z) = \frac{P_1(z)}{P_2(z)}$ where the real polynomials $P_1(z)$ and $P_2(z)$ have i_1 and i_2 roots, respectively in the interior of the circle $\mathcal{C}_\rho$ and no roots on the circle. Then

$$\Delta_0^\pi[\phi_Q(\theta)] = \pi\,(i_1 - i_2) = \Delta_{-1}^{+1}[\phi_{Q_C}(u)].$$

- Let $t_1, \cdots, t_k$ denote the real distinct zeros of $T(u,\rho)$ of odd multiplicity, for $u \in (-1,1)$, ordered as follows: $-1 < t_1 < t_2 < \cdots < t_k < +1$. Suppose also that $T(u,\rho)$ has p zeros at $u=-1$ and let $f^i(x_0)$ denote the i-th derivative to $f(x)$ evaluated at $x = x_0$.

THEOREM

Let $P(z)$ be a real polynomial with no roots on the circle $\mathcal{C}_\rho$ and suppose that $T(u,\rho)$ has p zeros at $u=-1$. Then the number of roots i of $P(z)$ in the interior of the circle $\mathcal{C}_\rho$ is given by

$$i = \frac{1}{2}\mathrm{Sgn}\left[T^{(p)}(-1,\rho)\right]\left(\mathrm{Sgn}\left[R(-1,\rho)\right] + 2\sum_{j=1}^{k}(-1)^j\mathrm{Sgn}\left[R\left(t_j,\rho\right)\right] + (-1)^{k+1}\mathrm{Sgn}\left[R(+1,\rho)\right]\right).$$

- The result derived above can now be extended to the case of rational functions. Let $Q(z) = \frac{P_1(z)}{P_2(z)}$ where $P_i(z), i = 1, 2$ are real rational functions.

- Tchebyshev representation of $Q(z)$ on the circle $\mathcal{C}_\rho$.
  Let $R(u,\rho), T(u,\rho)$ be defined by:
$$\begin{aligned} R(u,\rho) &= R_1(u,\rho)R_2(u,\rho) + (1-u^2)T_1(u,\rho)T_2(u,\rho) \\ T(u,\rho) &= T_1(u,\rho)R_2(u,\rho) - R_1(u,\rho)T_(u,\rho) \end{aligned}$$

- Suppose that $T(u,\rho)$ has p zeros at $u = -1$ and let $t_1 \cdots t_k$ denote the real distinct zeros of $T(u,\rho)$ of odd multiplicity ordered as $-1 < t_1 < t_2 < \cdots < t_k < +1$.

THEOREM

Let $Q(z) = \frac{P_1(z)}{P_2(z)}$ where $P_i(z), i = 1, 2$ are real polynomials with i_1 and i_2 zeros respectively inside the circle $\mathcal{C}_\rho$ and no zeros on it. Then

$$i_1 - i_2 = \frac{1}{2}\text{Sgn}\left[T^{(p)}(-1,\rho)\right]\left(\text{Sgn}\left[R(-1,\rho)\right] + 2\sum_{j=1}^{k}(-1)^j\text{Sgn}\left[R\left(t_j,\rho\right)\right] + (-1)^{k+1}\text{Sgn}\left[R(+1,\rho)\right]\right).$$

DIGITAL PI, PD AND PID CONTROLLERS

- For PI controllers,

$$\begin{aligned} C(z) &= K_P + K_I T \cdot \frac{z}{z-1} = \frac{(K_P + K_I T)\left(z - \frac{K_P}{K_I T + K_P}\right)}{z-1} \\ &= \frac{K_1 (z - K_2)}{z-1} \quad \text{where} \quad K_P = K_1 K_2, \quad K_I = \frac{K_1 - K_1 K_2}{T}. \end{aligned}$$

- For PD controllers,

$$\begin{aligned} C(z) &= K_P + \frac{K_D}{T} \cdot \frac{z-1}{z} = \frac{(K_P + \frac{K_D}{T})\left(z - \frac{\frac{K_D}{T}}{K_P + \frac{K_D}{T}}\right)}{z} \\ &=: \frac{K_1 (z - K_2)}{z} \quad \text{where} \quad K_P = K_1 - K_1 K_2, \quad K_D = K_1 K_2 T. \end{aligned}$$

- The general formula of a discrete PID controller, using backward differences to preserve causality,

$$\begin{aligned} C(z) &= K_P + K_I T \cdot \frac{z}{z-1} + \frac{K_D}{T} \cdot \frac{z-1}{z} =: \frac{K_2 z^2 + K_1 z + K_0}{z(z-1)} \quad \text{where} \\ K_P &= -K_1 - 2K_0, \quad K_I = \frac{K_0 + K_1 + K_2}{T}, \quad K_D = K_0 T. \end{aligned}$$

COMPUTATION OF THE STABILIZING SET

Constant Gain Stabilization

- Plant $G(z) = \dfrac{N(z)}{D(z)}$
- The closed-loop characteristic polynomial is

$$\delta(z) = D(z) + KN(z).$$

- Tchebyshev representations of D(z) and N(z)

$$\begin{aligned} D\left(e^{j\theta}\right) &= R_D(u) + j\sqrt{1-u^2}T_D(u) \\ N\left(e^{j\theta}\right) &= R_N(u) + j\sqrt{1-u^2}T_N(u), \end{aligned}$$

- Note also that $N\left(e^{-j\theta}\right) = R_D(u) - j\sqrt{1-u^2}T_D(u)$ and $N\left(z^{-1}\right) = \frac{N_r(z)}{z^l}$ where $N_r(z)$ is the *reverse polynomial* and l is the degree of $N(z)$.

- $\delta(z)N\left(z^{-1}\right) = D(z)N\left(z^{-1}\right) + KN(z)N\left(z^{-1}\right)$

- $$\begin{aligned} \left.\frac{\delta(z)N_r(z)}{z^l}\right|_{z=e^{j\theta}} &= \left(R_D(u) + j\sqrt{1-u^2}T_D(u)\right)\left(R_N(u) - j\sqrt{1-u^2}T_N(u)\right) \\ &\qquad + K\left[R_N^2(u) + \left(1-u^2\right)T_N^2(u)\right] \\ &= \underbrace{R_D(u)R_N(u) + \left(1-u^2\right)T_D(u)T_N(u) + K\left[R_N^2(u) + \left(1-u^2\right)T_N^2(u)\right]}_{R(K,u)} \\ &\qquad + j\sqrt{1-u^2}\underbrace{\left[T_D(u)R_N(u) - R_D(u)T_N(u)\right]}_{T(u)} \\ &= R(K,u) + j\sqrt{1-u^2}T(u). \end{aligned}$$

- The imaginary part of the above expression has been rendered independent of K as a result of multiplying $\delta(z)$ by $N(z^{-1})$.

parameter separation

Ready to apply the root counting formulas

Constant Gain Stabilization Algorithm

- Let $t_i, i = 1, 2 \cdots, k$ denote the real zeros of odd multiplicity of the fixed $T(u)$, for u in $(-1, +1)$ and set $t_0 = -1, t_{k+1} = +1$.
- Write $\mathrm{Sgn}\left[R(K, t_j)\right] = x_j, \qquad j = 0, 1, \cdots, k+1$
- Let i_δ, i_{N_r} denote the number of zeros of $\delta(z)$ and $N_r(z)$ inside the unit circle. For simplicity assume that $N(z)$ has no unit circle zeros and therefore neither does $N_r(z)$.

$$i_\delta + i_{N_r} - l = \frac{1}{2}\mathrm{Sgn}\left[T^{(p)}(-1)\right]$$
$$\cdot\left(\mathrm{Sgn}\left[R(K,-1)\right] + 2\sum_{j=1}^{k}(-1)^j \mathrm{Sgn}\left[R\left(K, t_j\right)\right] + (-1)^{k+1}\mathrm{Sgn}\left[R(K,+1)\right]\right).$$

Example

$$G(z) = \frac{z^4 + 1.93z^3 + 2.2692z^2 + 0.1443z - 0.7047}{z^5 - 0.2z^4 - 3.005z^3 - 3.9608z^2 - 0.0985z + 1.2311}.$$

- Then

$$\begin{aligned} R_D(u) &= -16u^5 - 1.6u^4 + 32.02u^3 - 6.3216u^2 - 13.9165u + 4.9919 \\ T_D(u) &= 16u^4 + 1.6u^3 - 24.02u^3 + 7.1216u + 3.9065 \\ R_N(u) &= 8u^4 - 7.72u^3 - 3.4616u^2 + 5.6457u - 1.9739 \\ T_N(u) &= -8u^3 + 7.72u^2 - 0.5384u - 1.7857 \end{aligned}$$

- and

$$\begin{aligned} T(u) &= T_D(u)R_N(u) - R_D(u)T_N(u) \\ &= -11.2752u^4 + 7.5669u^3 + 16.7782u^2 - 14.1655u + 1.203. \end{aligned}$$

- The roots of $T(u)$ of odd multiplicity and lying in $(-1, 1)$ are 0.0963 and 0.8358.

$$R(K,u) = 11.2752u^5 + 12.1307u^4 - 40.6359u^3 - 7.1779u^2 + 40.8322u$$
$$-16.8293 - 19.6615u - 5.4727$$
$$+K\left(-11.2752u^4 + 9.7262u^3 + 15.0696u^2 - 20.3653u + 7.085\right).$$

- Since $i_\delta = 5$ for stability, and $i_{N_r} = 2$ and $l = 4$, we must have:

$$\text{Sgn}\left[T^{(p)}(-1)\right]\left(\text{Sgn}[R(K,-1)] - 2\text{Sgn}[R(K,0.0963)] + 2\text{Sgn}[R(K,0.8358)] - \text{Sgn}[R(K,1)]\right) = 6$$

- Since $\text{Sgn}\left[T^{(p)}(-1)\right] = +1$, we have the only feasible string given by:

Sgn[R(K,-1)]	Sgn[R(K, 0.0963)]	Sgn[R(K, 0.8358)]	Sgn[R(K, 1)]
1	-1	1	-1

- This translates into the following set of inequalities:

$$\begin{aligned} R(K,-1) &= -23.348 + 21.5185K > 0 \Rightarrow K > 1.085 \\ R(K,0.0963) &= -12.998 + 5.2709K < 0 \Rightarrow K < 2.466 \\ R(K,0.8358) &= -0.9232 + 0.7673K > 0 \Rightarrow K > 1.2032 \\ R(K,1) &= -0.4050 + 0.2403K < 0 \Rightarrow K < 1.6854. \end{aligned}$$

- The closed loop system is stable for $1.2032 < K < 1.6854$.

- In this example, we have $x_j, j = 0, 1, 2, 3$. Each x_j may assume the value $+1$ or -1 since 0 is excluded as we are testing for stability. This leads to $2^4 = 16$ possible strings which may satisfy the signature requirement. In this example, only one string of the possible 16 satisfies the signature requirement.

Stabilization with PI Controllers

- Plant and Controller: $P(z) = \dfrac{N(z)}{D(z)}, \qquad C(z) = \dfrac{K_1\,(z-K_2)}{z-1}$
- The characteristic polynomial: $\delta(z) = (z-1)D(z) + K_1\,(z-K_2)\,N(z)$
- Writing the Tchebyshev representations of $D(z)$, $N(z)$ and $N\left(z^{-1}\right)$
- Then to achieve parameter separation, we calculate

$$\begin{aligned}\delta(z)N\left(z^{-1}\right)|_{u=-\cos\theta} &= \left(-u-1+j\sqrt{1-u^2}\right)\left(P_1(u)+j\sqrt{1-u^2}P_2(u)\right)\\ &\quad +jK_1\sqrt{1-u^2}P_3(u) - K_1\left(u+K_2\right)P_3(u)\end{aligned}$$

where

$$\begin{aligned}P_1(u) &= R_D(u)R_N(u) + \left(1-u^2\right)T_D(u)T_N(u)\\ P_2(u) &= R_N(u)T_D(u) - T_N(u)R_D(u)\\ P_3(u) &= R_N^2(u) + \left(1-u^2\right)T_N^2(u).\end{aligned}$$

$$\begin{aligned} \delta(z)N\left(z^{-1}\right)\Big|_{z=e^{j\theta},u=-\cos\theta} &= \left.\frac{\delta(z)N_r(z)}{z^l}\right|_{z=e^{j\theta},u=-\cos\theta} \\ &= R\left(u,K_1,K_2\right)+\sqrt{1-u^2}\,T\left(u,K_1\right) \end{aligned}$$

$$\text{where} \quad \begin{aligned} R\left(u,K_1,K_2\right) &= -(u+1)P_1(u)-\left(1-u^2\right)P_2(u)-K_1\left(u+K_2\right)P_3(u) \\ T\left(u,K_1\right) &= P_1(u)-(u+1)P_2(u)+K_1P_3(u). \end{aligned}$$

- For a fixed value of K_1, we calculate the real distinct zeros t_i of $T\left(u,K_1\right)$ of odd multiplicity for $u \in (-1,1)$: $-1 < t_1 < \cdots < t_k < +1$.
- Let i_δ, i_{N_r} be the number of zeros of $\delta(z)$ and $N_r(z)$ inside the unit circle, respectively, then we have

$$i_\delta + i_{N_r} - l = \tfrac{1}{2}\mathrm{Sgn}\left[T^{(p)}(-1)\right]\Big(\mathrm{Sgn}\left[R\left(-1,K_1,K_2\right)\right] \\ +2\textstyle\sum_{j=1}^{k}(-1)^j\mathrm{Sgn}\left[R\left(t_j,K_1,K_2\right)\right]+(-1)^{k+1}\mathrm{Sgn}\left[R\left(+1,K_1,K_2\right)\right]\Big).$$

Stabilization with PD Controllers

- Plant and Controller: $P(z) = \dfrac{N(z)}{D(z)}, \qquad C(z) = \dfrac{K_1 (z - K_2)}{z}$
- The characteristic polynomial: $\qquad \delta(z) = zD(z) + K_1 (z - K_2) N(z)$
- Consider

$$\delta(z) N\left(z^{-1}\right)\Big|_{z=e^{j\theta}, u=-\cos\theta} = R(u, K_1, K_2) + j\sqrt{1-u^2}\,T(u, K_1)$$

where

$$\begin{aligned} R(u, K_1, K_2) &= -uP_1(u) - \left(1-u^2\right) P_2(u) - K_1 (u + K_2) P_3(u) \\ T(u, K_1) &= K_1 P_3(u) + P_1(u) - uP_2(u). \end{aligned}$$

- Parameter separation has again been achieved, that is, K_1 appears only in the imaginary part and for fixed K_1 the real part is linear in K_2.
- Thus the application of the root counting formulas will yield linear inequalities in K_2, for fixed K_1.

STABILIZATION WITH PID CONTROLLERS

- PID Controller: $$C(z) = \frac{K_2 z^2 + K_1 z + K_0}{z(z-1)}$$

- The characteristic polynomial becomes
$$\delta(z) = z(z-1)D(z) + \left(K_2 z^2 + K_1 z + K_0\right) N(z)$$

- Multiplying the characteristic polynomial by $z^{-1}N\left(z^{-1}\right)$,
$$z^{-1}\delta(z)N\left(z^{-1}\right) = (z-1)D(z)N\left(z^{-1}\right) + \left(K_2 z + K_1 + K_0 z^{-1}\right) N(z)N\left(z^{-1}\right).$$

- Using the Tchebyshev representations, we have
$$\begin{aligned} z^{-1}\delta(z)N\left(z^{-1}\right) &= -(u+1)P_1(u) - \left(1-u^2\right) P_2(u) - \left[\left(K_0 + K_2\right) u - K_1\right] P_3(u) \\ &\quad + j\sqrt{1-u^2}\left[-(u+1)P_2(u) + P_1(u) + \left(K_2 - K_0\right) P_3(u)\right] \\ &= R\left(u, K_0, K_1, K_2\right) + j\sqrt{1-u^2}\, T\left(u, K_0, K_2\right). \end{aligned}$$

- Let $K_3 := K_2 - K_0$.

- Then $K_P = -K_1 - 2K_0, \quad K_I = \frac{K_0+K_1+K_2}{T}, \quad$ and $\quad K_D = K_0 T$.

- Hence we rewrite $R(u, K_0, K_1, K_2)$ and $T(u, K_0, K_2)$ as follows.

$$\begin{aligned} R(u, K_1, K_2, K_3) &= -(u+1)P_1(u) - \left(1-u^2\right) P_2(u) - \left[(2K_2 - K_3)u - K_1\right] P_3(u) \\ T(u, K_3) &= P_1(u) - (u+1)P_2(u) + K_3 P_3(u) \end{aligned}$$

- The parameter separation achieved : K_3 appears only in the imaginary part and K_1, K_2, K_3 appear linearly in the real part.

- Thus by applying root counting formulas to the rational function on the left, and imposing the stability requirement yields linear inequalities in the parameters for fixed K_3.

- The solution is completed by sweeping over the range of K_3 for which an adequate number of real roots t_k exist.

Example

- Plant: $G(z) = \dfrac{1}{z^2 - 0.25}$
- Then $R_D(u) = 2u^2 - 1.25, \quad T_D(u) = -2u, \quad R_N(u) = 1, \quad T_N(u) = 0$

  $P_1(u) = 2u^2 - 1.25, \quad P_2(u) = -2u, \quad P_3(u) = 1$
- Since $G(z)$ is of order 2 and $C(z)$, the PID controller, is of order 2, the number of roots of $\delta(z)$ inside the unit circle is required to be 4 for stability.
- From Theorem (Root counting for a real polynomial),

$$i_i - i_2 = \underbrace{(i_\delta + i_{N_r})}_{i_1} - \underbrace{(l+1)}_{i_2}$$

  where i_δ and i_{N_r} are the numbers of roots of $\delta(z)$ and the reverse polynomial of $N(z)$ inside the unit circle, respectively and l is the degree of $N(z)$.

- Since the required i_δ is 4, $i_{N_r} = 0$, and $l = 0$, $i_1 - i_2$ is required to be 3.
- To illustrate the example in detail, we first fix $K_3 = 1.3$.
- Then the real roots of $T(u, K_3)$ in $(-1, 1)$ are -0.4736 and -0.0264.
- Furthermore, $\text{Sgn}[T(-1)] = 1$, $i_1 - i_2 = 3$ requires that:

$$\frac{1}{2}\text{Sgn}[T(-1)]\Big(\text{Sgn}[R(-1)] - 2\text{Sgn}[R(-0.4736)] + 2\text{Sgn}[R(-0.0264)] - \text{Sgn}[R(1)]\Big) = 3$$

- We have only one valid sequence satisfying the above equation,

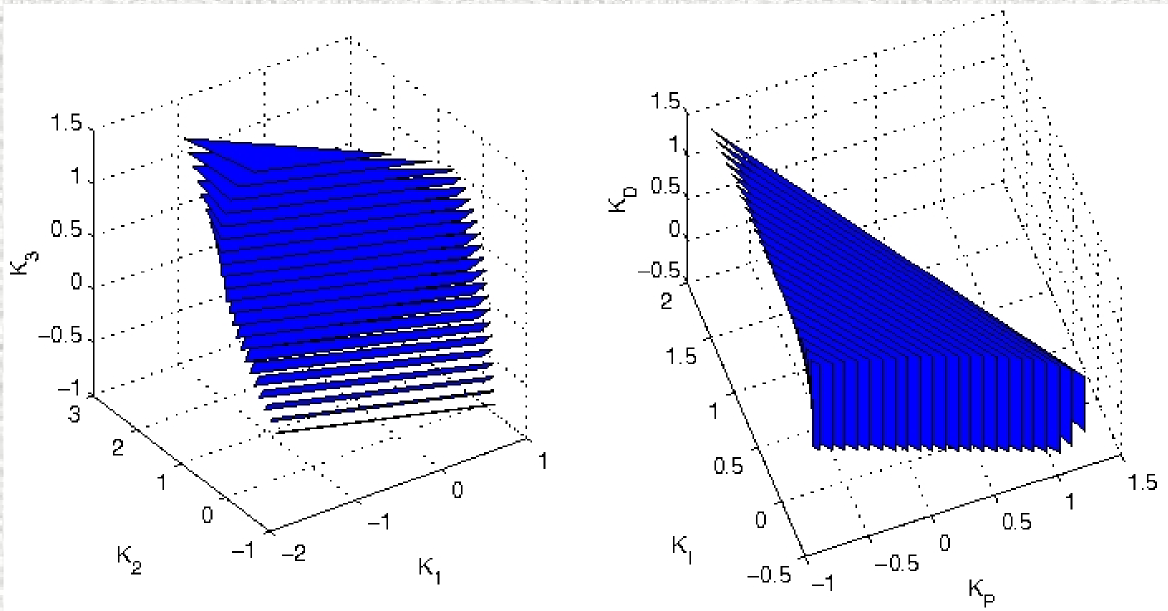
Sgn[R(-1)]	Sgn[R(-0.4736)]	Sgn[R(-0.0264)]	Sgn[R(1)]	$2(i_1 - i_2)$
1	-1	1	-1	6

- From this valid sequence, we have the following set of linear inequalities.

$$\begin{array}{rcl} -1.3 + K_1 + 2K_2 & > & 0 \\ -0.9286 + K_1 + 0.9472 & < & 0 \\ 1.1286 + K_1 + 0.0528K_2 & > & 0 \\ -0.2 + K_1 - 2K_2 & < & 0. \end{array}$$

$$\begin{bmatrix} K_P \\ K_I \\ K_D \end{bmatrix} = \begin{bmatrix} -2 & -1 & 0 \\ \frac{1}{T} & \frac{1}{T} & \frac{1}{T} \\ T & 0 & 0 \end{bmatrix} \begin{bmatrix} K_0 \\ K_1 \\ K_2 \end{bmatrix} = \begin{bmatrix} -2 & -1 & 0 \\ \frac{1}{T} & \frac{1}{T} & \frac{1}{T} \\ T & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 \\ K_3 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -2 & 2 \\ \frac{1}{T} & \frac{2}{T} & -\frac{1}{T} \\ 0 & T & -T \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 \\ K_3 \end{bmatrix}.$$



Stability regions in (K_1, K_2, K_3) space (left) and (K_P, K_I, K_D) space (right)

Maximally Deadbeat Control

- The design scheme attempts to place the closed loop poles in a circle of *minimum* radius ρ. Let $\mathcal{S}_\rho$ denote the set of PID controllers achieving such a closed loop root cluster.

- We show below how $\mathcal{S}_\rho$ can be computed for fixed ρ. The minimum value of ρ can be found by determining the value ρ^* for which $\mathcal{S}_{\rho^*} = \phi$ but $\mathcal{S}_\rho \neq \phi, \rho > \rho^*$.

- PID Controller: $C(z) = \dfrac{K_2 z^2 + K_1 z + K_0}{z(z-1)}$

- The characteristic equation
$$\delta(z) = z(z-1)D(z) + \left(K_2 z^2 + K_1 z + K_0\right) N(z).$$

- Note that
$$\begin{aligned} D(z)|_{z=-\rho u + j\rho\sqrt{1-u^2}} &= R_D(u,\rho) + j\sqrt{1-u^2}\,T_D(u,\rho) \\ N(z)|_{z=-\rho u + j\rho\sqrt{1-u^2}} &= R_N(u,\rho) + j\sqrt{1-u^2}\,T_N(u,\rho) \end{aligned}$$

$$\begin{aligned} N\left(\rho^2 z^{-1}\right)\Big|_{z=-\rho u+j\rho\sqrt{1-u^2}} &= N(z)|_{z=-\rho u-j\rho\sqrt{1-u^2}} \\ &= R_N(u,\rho) - j\sqrt{1-u^2}T_N(u,\rho). \end{aligned}$$

- We evaluate

$$\rho^2 z^{-1}\delta(z)N\left(\rho^2 z^{-1}\right) = \rho^2 z^{-1}\underbrace{\left[z(z-1)D(z) + \left(K_2 z^2 + K_1 z + K_0\right)N(z)\right]}_{\delta(z)} N\left(\rho^2 z^{-1}\right)$$

over the circle $\mathcal{C}_\rho$

$$\begin{aligned} & \rho^2 z^{-1}\delta(z)N\left(\rho^2 z^{-1}\right)\Big|_{z=-\rho u+j\rho\sqrt{1-u^2}} \\ = \;& -\rho^2(\rho u+1)P_1(u,\rho) - \rho^3\left(1-u^2\right)P_2(u,\rho) - \left[\left(K_0 + K_2\rho^2\right)\rho u - K_1\rho^2\right]P_3(u,\rho) \\ & + j\sqrt{1-u^2}\left[\rho^3 P_1(u,\rho) - \rho^2(\rho u+1)P_2(u,\rho) + \left(K_2\rho^2 - K_0\right)\rho P_3(u,\rho)\right] \end{aligned}$$

where

$$\begin{aligned} P_1(u,\rho) &= R_D(u,\rho)R_N(u,\rho) + \left(1-u^2\right)T_D(u,\rho)T_N(u,\rho) \\ P_2(u,\rho) &= R_N(u,\rho)T_D(u,\rho) - T_N(u,\rho)R_D(u,\rho) \\ P_3(u,\rho) &= R_N^2(u,\rho) + \left(1-u^2\right)T_N^2(u,\rho). \end{aligned}$$

- By letting $K_3 := K_2\rho^2 - K_0$,

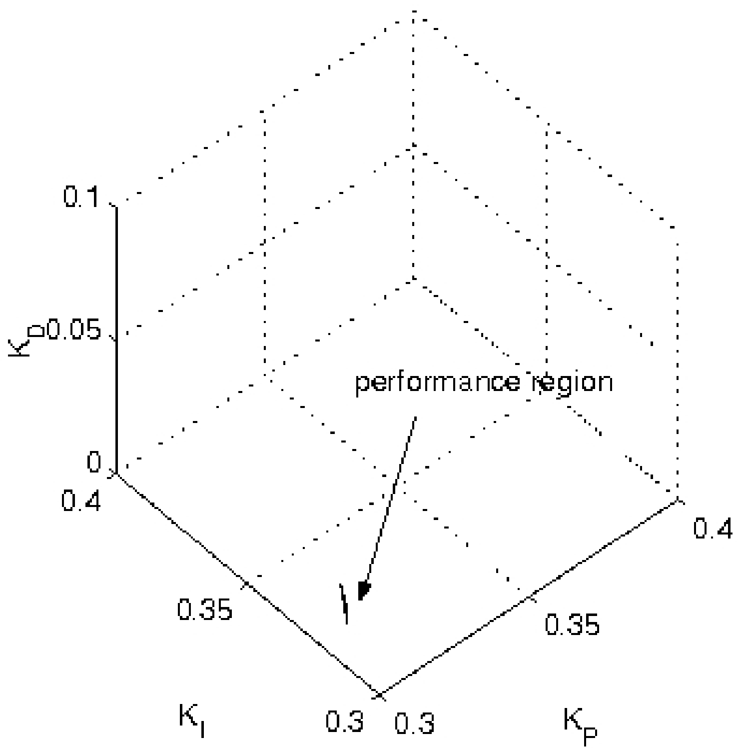
- we have

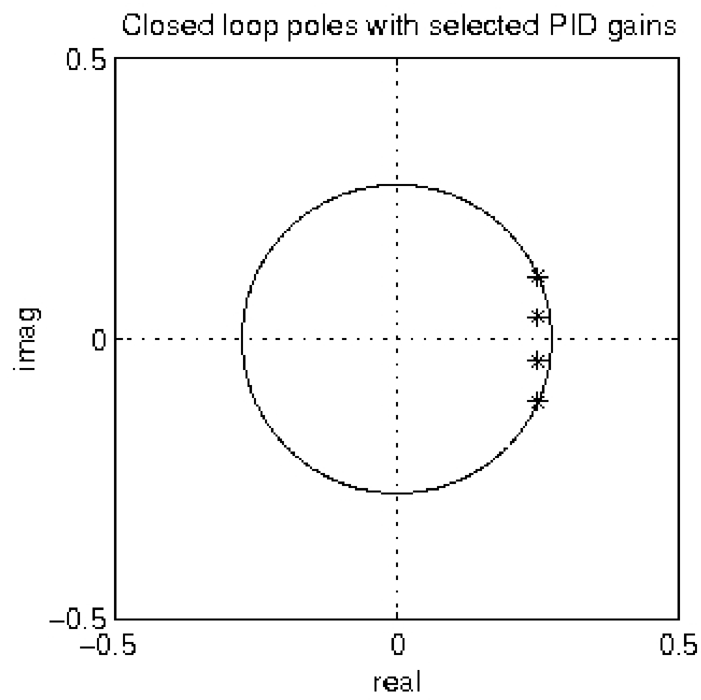
$$
\begin{aligned}
& \rho^2 z^{-1}\delta(z)N\left(\rho^2 z^{-1}\right)\big|_{z=-\rho u+j\rho\sqrt{1-u^2}} \\
= & -\rho^2(\rho u+1)P_1(u,\rho) - \rho^3\left(1-u^2\right)P_2(u,\rho) - \left[\left(2K_2\rho^2 - K_3\right)\rho u - K_1\rho^2\right]P_3(u,\rho) \\
& +j\sqrt{1-u^2}\left[\rho^3 P_1(u,\rho) - \rho^2(\rho u+1)P_2(u,\rho) + K_3\rho P_3(u,\rho)\right].
\end{aligned}
$$

- Fix K_3, use the root counting formulas, develop linear inequalities in K_2, K_3 and sweep over the requisite range of K_3. This procedure is then performed as ρ decreases until the set of stabilizing PID parameters just disappears.

Example

- We consider the same plant used in the previous example.
- Left figure shows the stabilizing set in the PID gain space at $\rho = 0.275$.





- For a smaller value of ρ, the stabilizing region in PID parameter space disappears. This means that there is no PID controller available to push all closed loop poles inside a circle of radius smaller than 0.275.

- From this we select a point inside the region that is

$$K_0 = 0.0048, \quad K_1 = -0.3195, \quad K_2 = 0.6390, \quad K_3 = 0.0435.$$

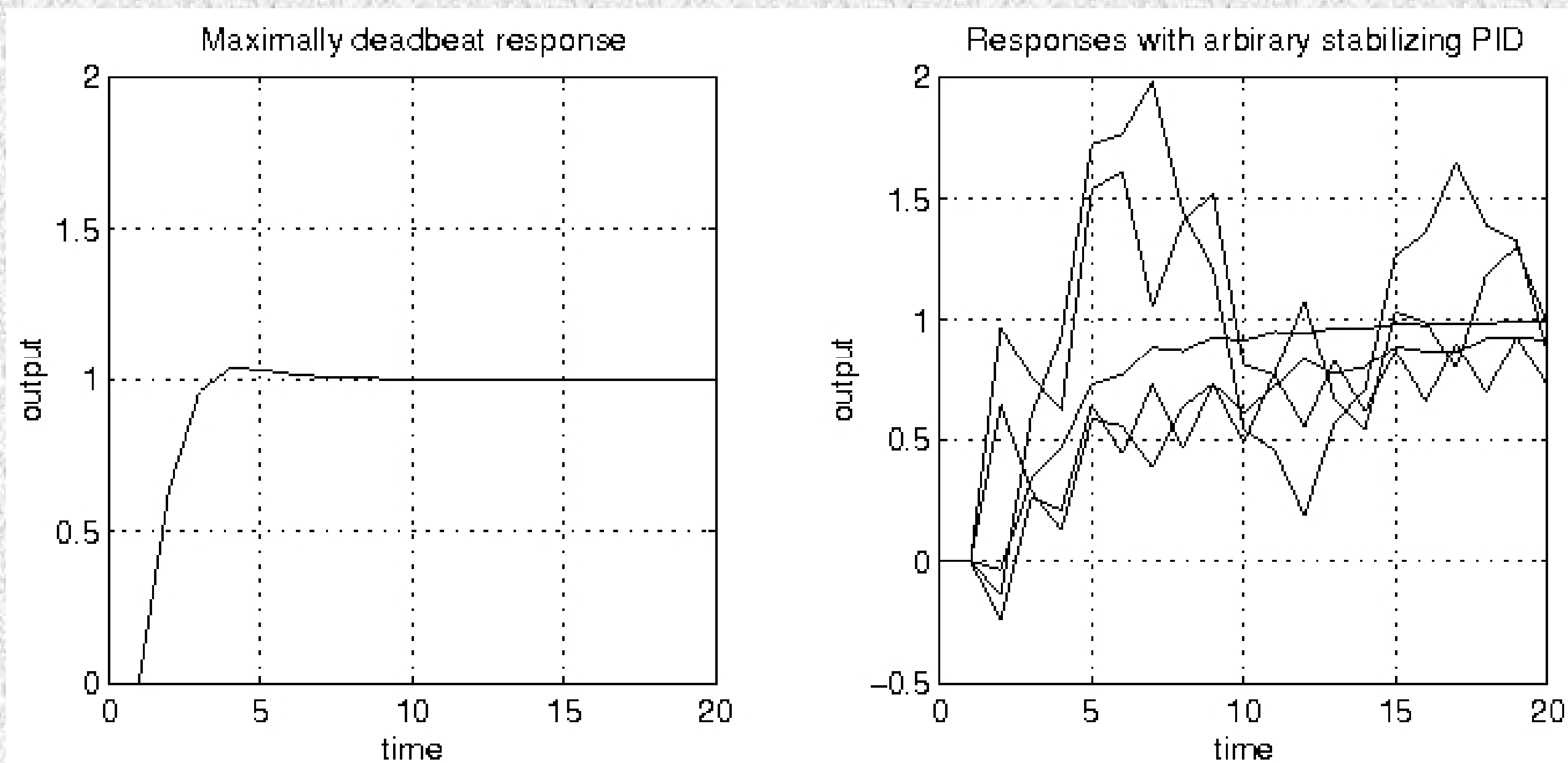
- From the relationship between parameters, we have

$$\begin{bmatrix} K_P \\ K_I \\ K_D \end{bmatrix} = \begin{bmatrix} -1 & -2\rho^2 & 2 \\ \frac{1}{T} & \frac{\rho^2}{T} + \frac{1}{T} & -\frac{1}{T} \\ 0 & \rho^2 T & -T \end{bmatrix} \begin{bmatrix} K1 \\ K2 \\ K3 \end{bmatrix} = \begin{bmatrix} 0.3099 \\ 0.3243 \\ 0.0048 \end{bmatrix}$$

- Right figure shows the closed loop poles that lie inside the circle of radius $\rho = 0.275$. The roots are:

$$0.2500 \pm j0.1118 \qquad \text{and} \qquad 0.2500 \pm j0.0387.$$

- We select several sets of stabilizing PID parameters from the set obtained in the previous example (i.e., $\rho = 1$) and compare the step responses between them.



Maximum Delay Tolerance Design

- Finding the maximum values of L^* such that the stabilizing PID gain set that simultaneously stabilizes the set of plants

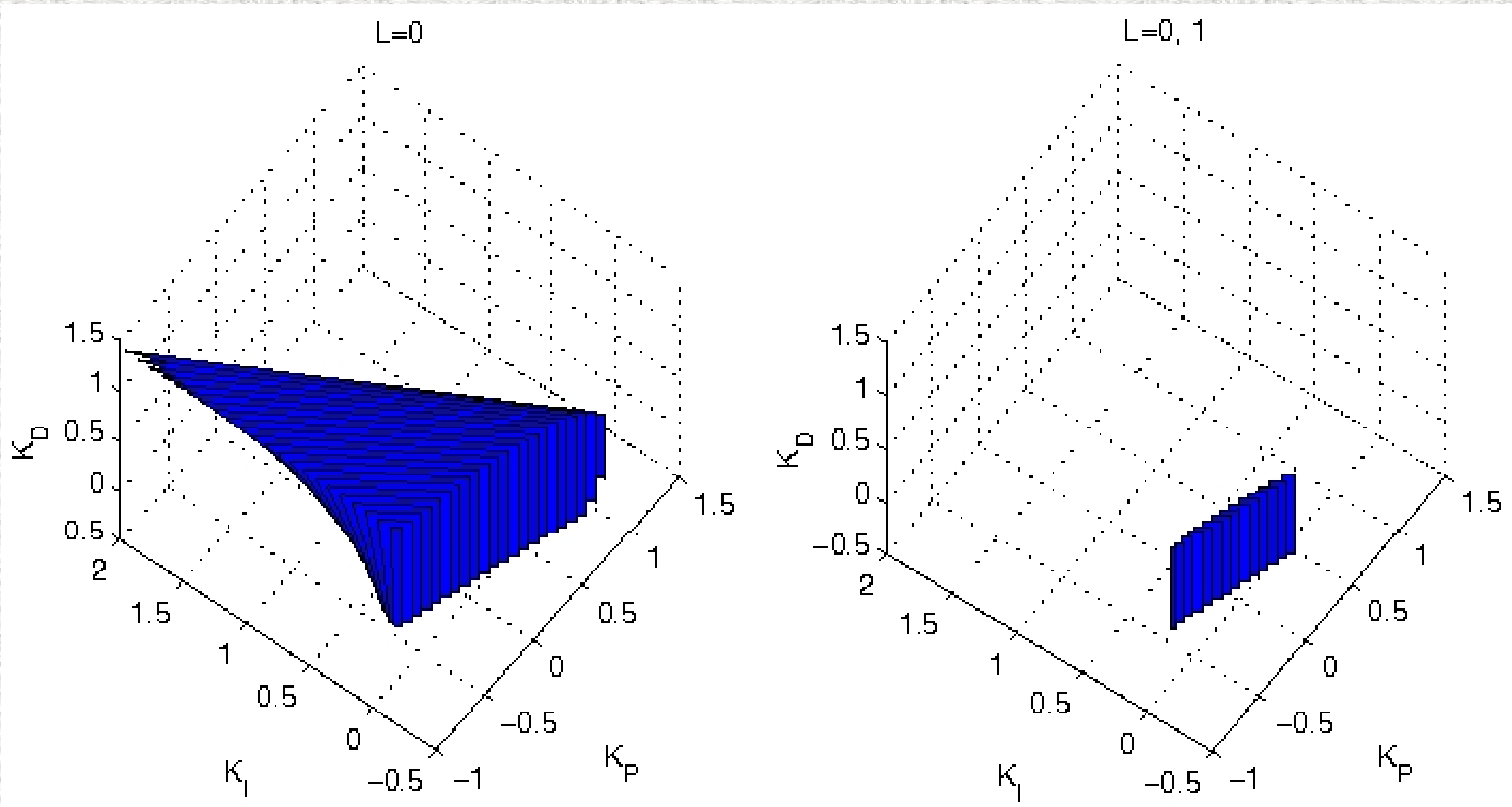
$$z^{-L}G(z) = \frac{N(z)}{z^L D(z)}, \qquad \text{for } L = 0, 1, \cdots, L^*$$

is not empty.

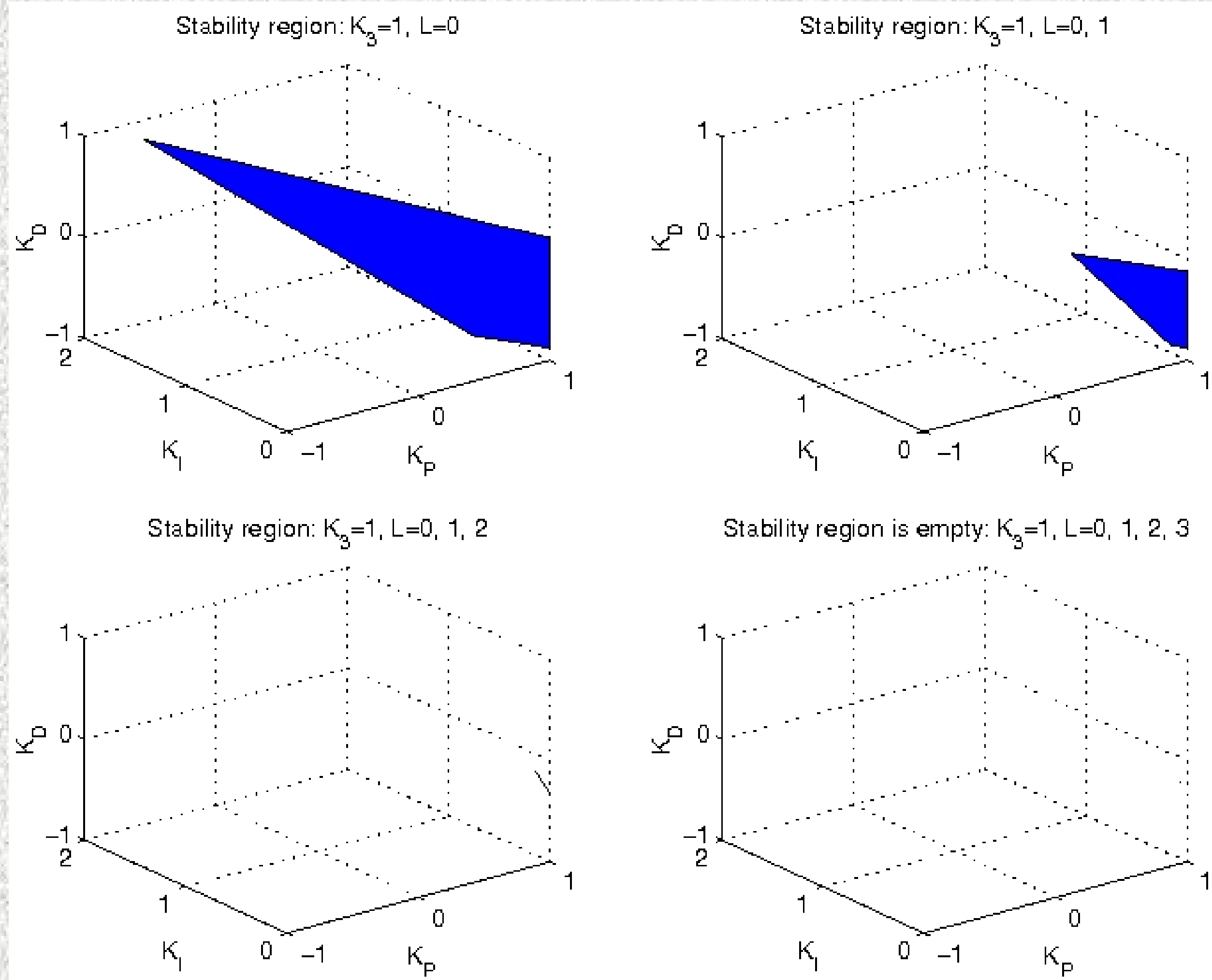
- Let $\mathcal{S}_i$ be the set of PID gains that stabilizes the plant $z^{-i}G(z)$. Then

$$\cap_{i=0}^{L}\mathcal{S}_i \ \text{ stabilizes } \ z^{i}G(z) \text{ for all } i = 0, 1, \cdots, L.$$

Example



- The right figure shows the stabilizing PID gains when $L = 0, 1$. As seen in the figure, the size of the set is reduced as the delay increases.



- In many systems, the set disappears for a large value of L^*. This is the maximum delay that can be stabilized by any PID controllers.