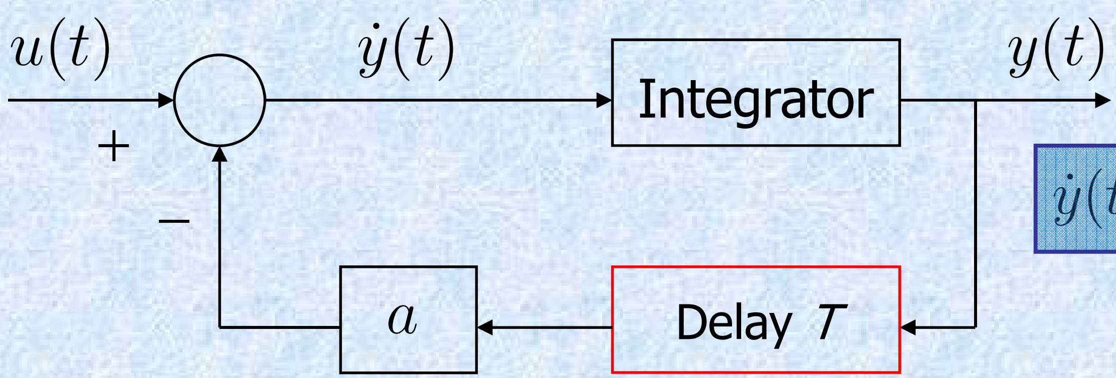
PID Controllers for Systems with Time-Delay

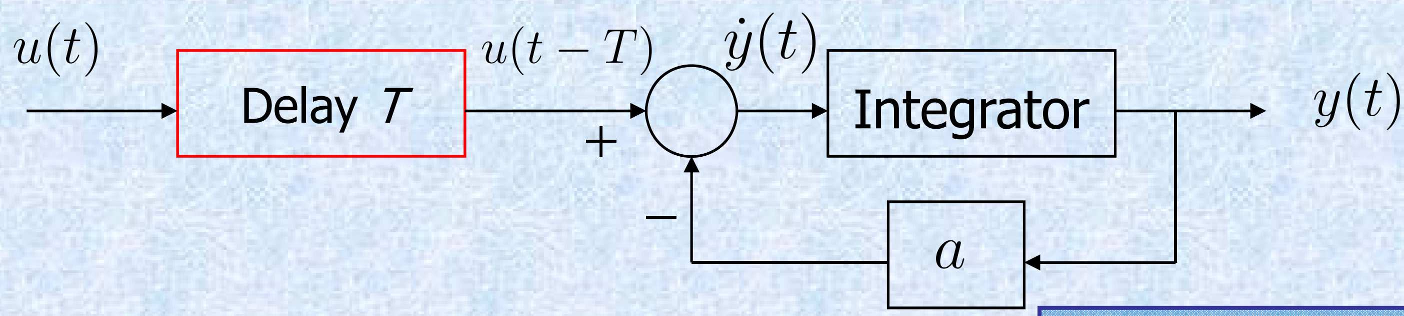
General Considerations

CHARACTERISTIC EQUATIONS FOR DEALY SYSTEMS

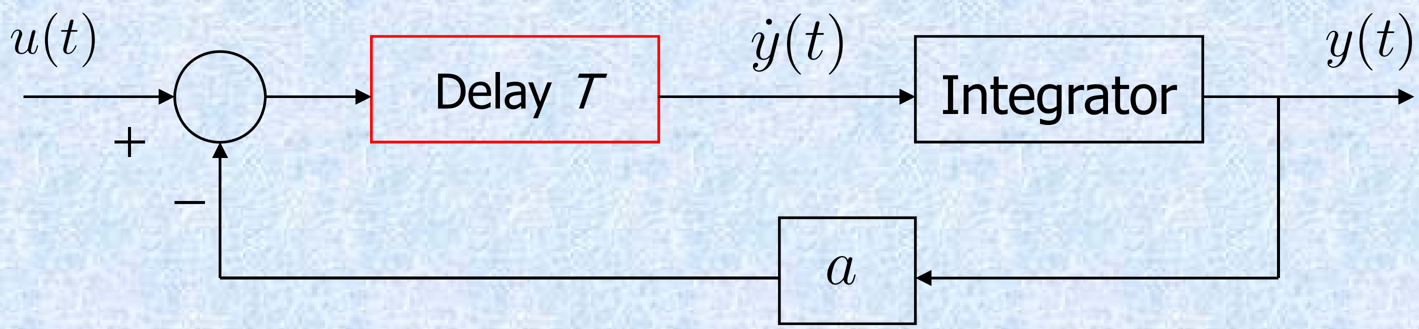
$$y(t) = u(t - T)$$



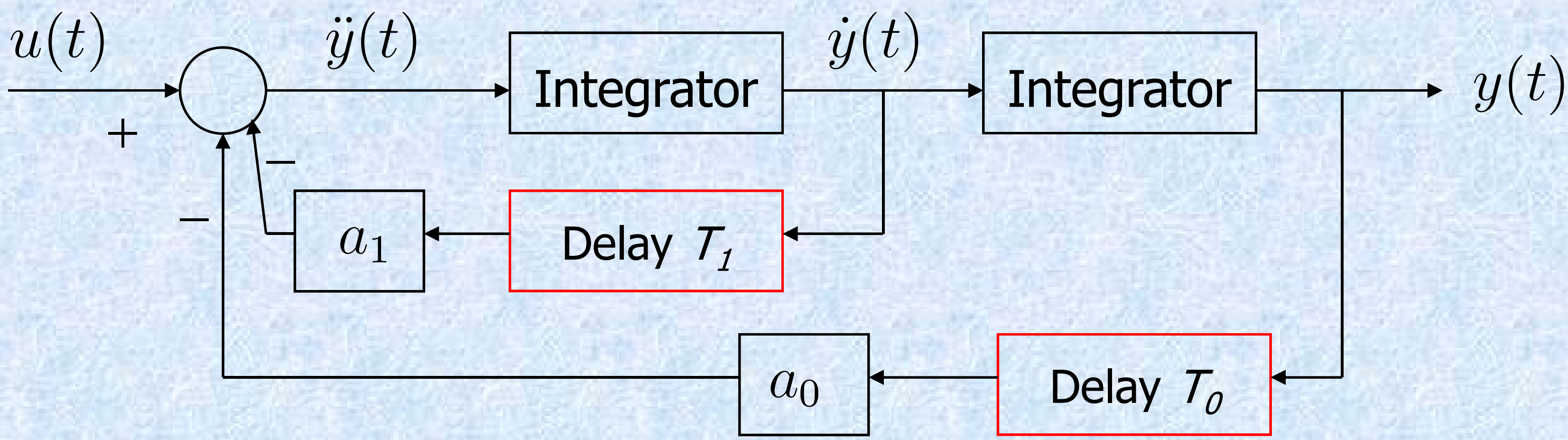
$$\dot{y}(t) + ay(t - T) = u(t)$$



$$\dot{y}(t) + ay(t) = u(t-T)$$



$$\dot{y}(t) = -ay(t-T) + u(t-T)$$



$$\ddot{y}(t) + a_1 \dot{y}(t - T_1) + a_0 y(t - T_0) = u(t)$$

Let $y(t) = x_1(t), \quad \dot{y}(t) = x_2(t)$

Then
$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -a_0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t - T_0) \\ x_2(t - T_0) \end{bmatrix}$$
$$+ \begin{bmatrix} 0 & 0 \\ 0 & -a_1 \end{bmatrix} \begin{bmatrix} x_1(t - T_1) \\ x_2(t - T_1) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \; .$$

Stability of Delay Systems

- Let $y(t)=e^{st}$ be a proposed solution of

$$\ddot{y}(t) + a_1\dot{y}(t-T_1) + a_0 y(t-T_0) = 0$$

- Then we have $\left(s^2 + a_1 e^{-sT_1} s + a_0 e^{-sT_0}\right) e^{st} \equiv 0$

  so that "s" must satisfy $\boxed{s^2 + a_1 s e^{-sT_1} + a_0 e^{-sT_0} = 0}$

Characteristic equation of the delay system.

- The location of its zeros determine the stability of the system.

- If any roots lie in the closed RHP, the system is unstable as the solution grows without bound.

- Consider a LTI system with ℓ distinct delays,

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{l} A_i x(t - T_i) + Bu(t)$$

- The corresponding characteristic equation is

$$\delta(s) := \det\left(sI - A_0 - \sum_{i=1}^{l} e^{-sT_i} A_i\right) = P_0(s) + \sum_{k=1}^{m} P_k(s) e^{-L_k s}$$

and $$P_0(s) = s^n + \sum_{i=0}^{n-1} a_i s^i, \qquad P_k(s) = \sum_{i=0}^{n-1} (b_k)_i s^i$$

- (Retarded Delay Systems)

$$\ddot{y}(t) + a_1 \dot{y}(t - T_1) + a_0 y(t - T_0) = u(t)$$

- (Neutral Delay System)

$$\ddot{y}(t - T_2) + a_1 \dot{y}(t - T_1) + a_0 y(t - T_0) = u(t)$$

Roots of Characteristic Equations

- Retarded Systems: There can only be a finite number of RHP roots. The stability of retarded systems is equivalent to the absence of closed RHP roots.

- The fact that retarded systems have a finite number of RHP roots means that one can count the number of roots crossing into the RHP through the stability boundary and keep track of the number of RHP roots as some parameter vary.

- Neutral Systems: Certain root chains can approach the imaginary axis from the LHP and thus destroy stability.

- If delays are multiples of a common delay, we have

$$\delta(s) = a_0(s) + a_1(s)e^{-\tau s} + a_2(s)e^{-2\tau s} + \cdots + a_k(s)e^{-k\tau s}$$

THE PADE APPROXIMATION AND ITS LIMITATIONS

$$e^{-sL} \cong \frac{N_r(sL)}{D_r(sL)}$$

where

$$\begin{aligned} N_r(sL) &= \sum_{k=0}^{r} \frac{(2r-k)!}{k!(r-k)!}(-sL)^k \\ D_r(sL) &= \sum_{k=0}^{r} \frac{(2r-k)!}{k!(r-k)!}(sL)^k \end{aligned}$$

For example, the 3rd order Pade approximation is given by

$$\frac{N_3(sL)}{D_3(sL)} = \frac{-L^3s^3 + 12L^2s^2 - 60Ls + 120}{L^3s^3 + 12L^2s^2 + 60Ls + 120}$$

PID Stabilization of a Delay Systems Using a 1st Order Pade Approximation (An Example)

- 1st Order Pade approximation $$e^{-sL} \cong \frac{2-Ls}{2+Ls}$$

- Plant $$G(s) = \left[\frac{k}{Ts+1}\right] e^{-sL} \cong \left[\frac{k}{(Ts+1)}\right] \left(\frac{(-Ls+2)}{(Ls+2)}\right)$$

- With the PID controller (k_p,k_i,k_d), the closed-loop characteristic polynomial becomes

$$\begin{aligned} \delta(s,k_p,k_i,k_d) &= s(Ts+1)(Ls+2) + (k_i + k_p s + k_d s^2)(k)(-Ls+2) \\ &= (Ts^2+s)(Ls+2) + (k_d' s^2 + k_i')(-Ls+2) + k_p' s(-Ls+2) \end{aligned}$$

where $k_d' = kk_d, k_i' = kk_i, k_p' = kk_p.$

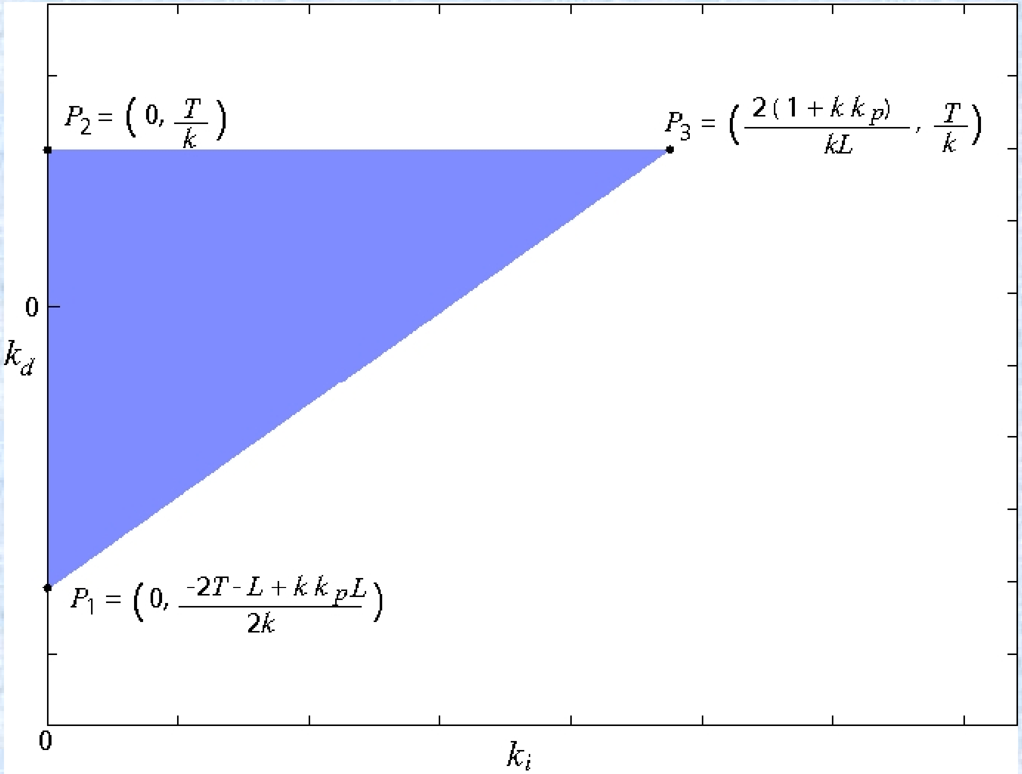
- Using the PID Design Algorithm, we have

$$
\begin{cases}
k_i > 0 \\
k_i - \left[\dfrac{4(1+kk_p)}{L(4T+L-kk_pL)}\right] k_d < \dfrac{2(1+kk_p)(2T+L-kk_pL)}{kL(4T+L-kk_pL)} \\
k_d < \dfrac{T}{k}
\end{cases}
$$

and

$$
-\frac{1}{k} < k_p < \frac{1}{k}\left(1+\frac{4T}{L}\right)
$$

For a fixed k_p, it becomes the set of linear inequalities in terms of k_i, k_d and can be solved by LP.



The stabilizing set of (k_i,k_d) values for a fixed k_p.

Question: Does the 1st order Pade approximation accurately capture the actual set of stabilizing PID parameters for the original time-delay system?

Next Example

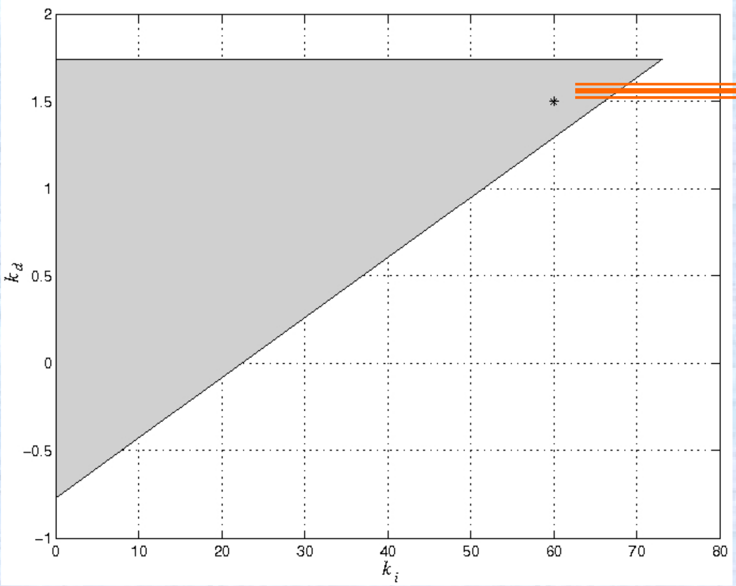
Example

- Plant $$G(s) = \left[\frac{1.6667}{1+2.9036s}\right] e^{-0.2475s}$$

- Plant with the 1st order Pade approximation

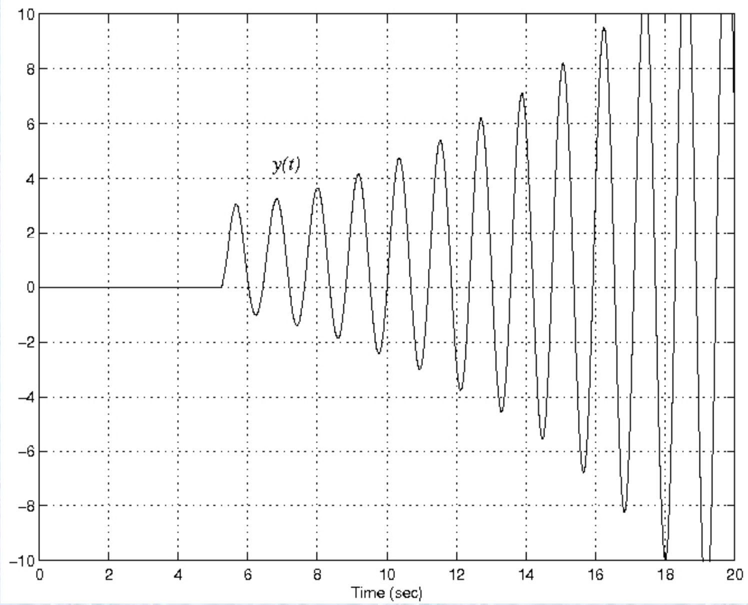
$$G_m(s) = \frac{1.6667}{(1+2.9036s)} \frac{(-0.1238s+1)}{(0.1238s+1)}$$

- Compute the entire stabilizing PID parameter values.



$$\begin{aligned} k_p &= 8.4467 \\ k_i &= 60 \\ k_d &= 1.5 \end{aligned}$$

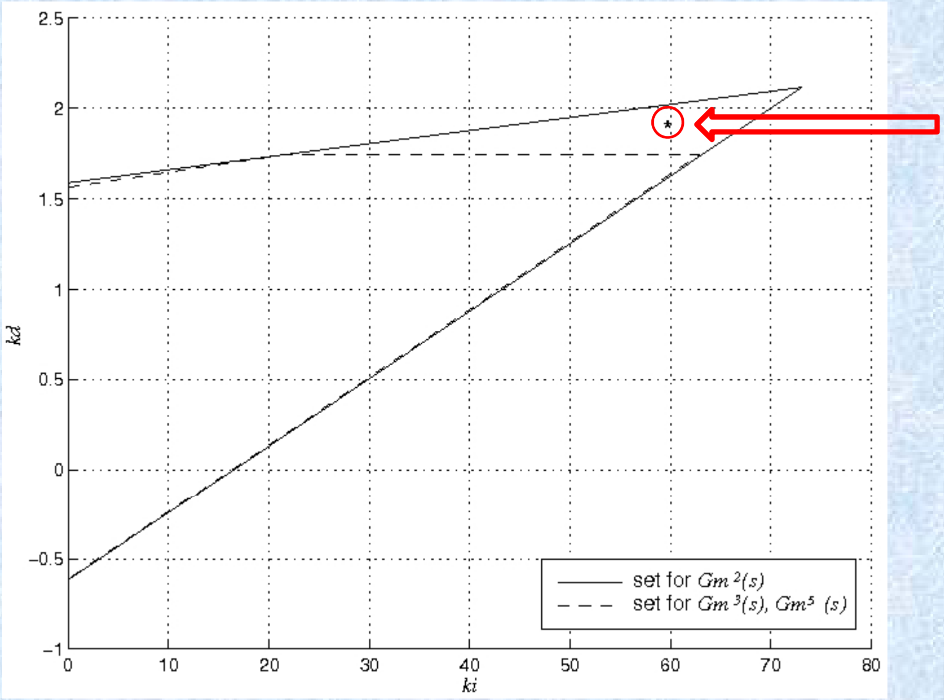
Time-response of the closed-loop system



The stabilizing (k_i,k_d) values at k_p=8.4467

Showing unstable behavior!

- Tried with the 2nd, 3rd, and 5th order Pade approximation

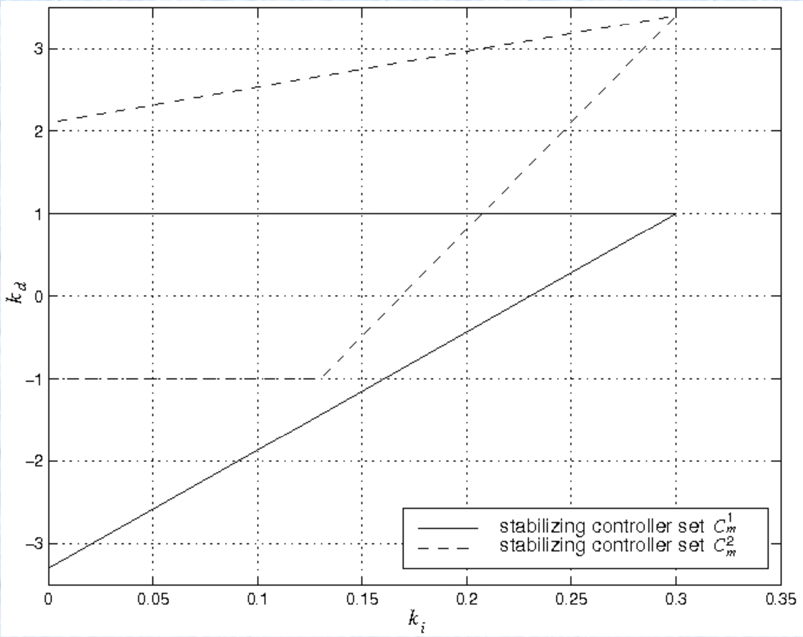


- While the 2nd order Pade approximation fails to capture the actual stabilizing set, the 3rd and 5th order Pade approximations apparently do a better job.

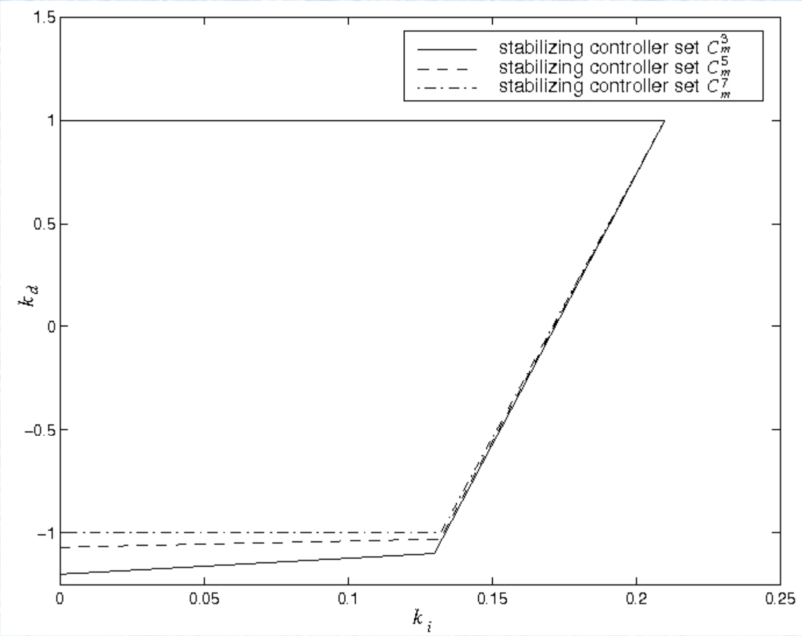
Example with large delay

- Plant $$G(s) = \left[\frac{1}{1+s}\right] e^{-10s}$$

- Approximate the time-delay term using the 1st, 2nd, 3rd, 5th, 7th, and 9th order Pade approximations



1st and 2nd order approximations



3rd, 5th, 7th, and 9th order approximations

Observations

- For small values of the time-delay, the approximate sets easily converge to the possible true sets. However, the convergence becomes more difficult as the value of the time-delay increases.
- The convergence of the approximate set to a possible true set improves with increased order of the Pade approximation.

- The Pade approximation is not a satisfactory tool for ensuring the stability of the resulting control design.
- It is not a priori clear as to what order of the approximation will yield a stabilizing set of parameters accurately approximating the true set.

Pontryagin's results

THE HERMITE-BIEHLER THEOREM FOR QUASI-POLYNOMIALS

Let $f(s,t)$ be a polynomial in two variables with real or complex coefficients defined as follows:

$$f(s,t) = \sum_{h=0}^{M} \sum_{k=0}^{N} a_{hk} s^h t^k$$

Definition

$f(s,t)$ is said to have a principal term if there exists a nonzero coefficient a_{hk} where both indices have maximal values. Without loss of generality, we will denote the principal term as $a_{MN} s^M t^N$. This means that for each other term $a_{hk} s^h t^k$, for $a_{hk} \neq 0$, we have either $M > h, N > k$; or $M = h, N > k$; or $M > h, N = k$.

Example $f(s,t) = 3s + t^2$ does not have a principal term but the polynomial $f(s,t) = s^2 + t + 2s^2 t$ does.

Theorem (Pontryagin)

If the polynomial $f(s,t)$ does not have a principal term, then the function $F(s) = f(s, e^s)$ has an infinite number of zeros with arbitrarily large positive real parts.

If $f(s,t)$ does have a principal term, the main result of Pontryagin is to show that the Hermite-Biehler Theorem extends to the class of functions $F(s) = f(s, e^s)$.

Study of the zeros of functions of the form *g(s,cos(s),sin(s))*

- Let *g(s,u,v)* be a polynomial with real coefficients:

$$g(s,u,v) = \sum_{h=0}^{M} \sum_{k=0}^{N} s^h \phi_h^{(k)}(u,v)$$

$\phi_h^{(k)}(u,v)$ is a polynomial of degree *k*, homogeneous in *u* and *v*.

- Assume that $\phi_h^{(k)}(u,v)$ is not divisible by $u^2 + v^2$:

$$\phi_h^{(k)}(1, \pm j) \neq 0$$

- Let $\phi^{*(N)}(u,v) = \sum_{k=0}^{N} \phi_M^{(k)}(u,v)$

the coefficient of s^M

- Consider $G(s) = g(s, \cos(s), \sin(s))$

- Let $\Phi^{*(N)}(s) := \phi^{*(N)}(\cos(s), \sin(s))$

THEOREM

Let *g(s,u,v)* be a polynomial with principal term given by $s^M \phi_M^{(N)}(u,v)$. If η is such that $\Phi^{*(N)}(\eta + j\omega)$ does not take the value zero for real ω, then starting from some sufficiently large value of l, the function *G(s)* will have exactly $4lN + M$ zeros in the strip

$$-2l\pi + \eta \leq \mathrm{Re}[s] \leq 2l\pi + \eta.$$

Thus for the function *G(s)* to have only real roots, it is necessary and sufficient that in the interval

$$-2l\pi + \eta \leq \mathrm{Re}[s] \leq 2l\pi + \eta,$$

it has exactly $4lN + M$ real roots starting with some sufficiently large l.

- Consider

$$f(s,t)=\sum_{h=0}^{M}\sum_{k=0}^{N}a_{hk}s^h t^k = s^M \underbrace{X^{*(N)}(t)} + \sum_{h=0}^{M-1}\sum_{k=0}^{N}a_{hk}s^h t^k$$

$$X^{*(N)}(t)=\sum_{k=0}^{N}a_{Mk}t^k$$

Definition

Let $F(s)=f(s,e^s)$. where *f(s,t)* is a polynomial with a principal term, and

$$F(j\omega)=F_r(\omega)+jF_i(\omega)$$

Let ω_{r1}, ω_{r2}, ω_{r3}, … denote the real roots of $F_r(\omega)$, and let ω_{i1}, ω_{i2}, ω_{i3}, … denote the real roots of $F_i(\omega)$, both arranged in ascending order of magnitude. The we say that the roots of $F_r(\omega)$ and $F_i(\omega)$ interlace if they satisfy the following property:

$$\omega_{r1}<\omega_{i1}<\omega_{r2}<\omega_{i2}<\cdots$$

THEOREM (HB Theorem to quasi-polynomial)

If all the roots of $F(s)$ lie in the open LHP, then the roots of $F_r(\omega)$ and $F_i(\omega)$ are real, simple, interlacing, and

$$F_i^{'}(\omega)F_r(\omega) - F_i(\omega)F_r^{'}(\omega) > 0 \qquad (*)$$

for each $\omega \in (-\infty, \infty)$, where $F_r^{'}(\omega)$ and $F_i^{'}(\omega)$ denote the first derivative with respect to ω of $F_r(\omega)$ and $F_i(\omega)$, respectively. Moreover, in order that all the roots of $F(s)$ lie in the open LHP, it is sufficient that one of the following conditions be satisfied:

1. All the roots of $F_r(\omega)$ and $F_i(\omega)$ are real, simple, and interlacing and the inequality (*) is satisfied for at least one value of ω;
2. All the roots of $F_r(\omega)$ are real and for each root , (*) is satisfied, i.e., $F_i(\omega_r)F_r^{'}(\omega_r) < 0;$
3. All the roots of $F_i(\omega)$ are real and for each root, (*) is satisfied, i.e., $F_i^{'}(\omega_i)F_r(\omega_i) > 0.$

THEOREM

If the function $X^{*(N)}(e^s)$ has roots in the open RHP, then the function *F(s)* has an unbounded set of zeros in the open RHP. If all the zeros of the function $X^{*(N)}(e^s)$ lie in the open LHP, then the function *F(s)* can only have a bounded set of zeros in the open RHP.

Application to Control Theory

Classes of Quasi-polynomials:

Retarded-type (or delay-type) Quasi-polynomials: This class consists of quasi-polynomials whose asymptotic chains go deep into the open LHP.

Neutral-type quasi-polynomials: This class consists of quasi-polynomials that along with delay-type chains contain at least one asymptotic chain of roots in a vertical strip of the complex plane.

Forestall-type quasi-polynomials: This class consists of quasi-polynomials with at least one asymptotic chain that goes deep into the open RHP.

Definition

A delay-type quasi-polynomial is said to be stable iff all its roots have negative real parts.

Definition

A neutral-type quasi-polynomial is said to be stable if there exists a positive number σ such that the real parts of all its roots are less than $-\sigma$.

THEOREM

Let $\delta^*(s) = e^{sL_m}d(s) + e^{s(L_m - L_1)}n_1(s) + e^{s(L_m - L_2)}n_2(s) + \cdots + n_m(s)$

and write $\delta^*(j\omega) = \delta_r(\omega) + j\delta_i(\omega)$.Under the following conditions

$$\text{(A1)} \quad \deg[d(s)] = q \text{ and } \deg[n_i(s)] \le q \text{ for } i = 1, 2, ..., m;$$
$$\text{(A2)} \quad 0 < L_1 < L_2 < \cdots < L_m$$

$\delta^*(s)$ is stable iff

1. $\delta_r(\omega)$ and $\delta_i(\omega)$ have only simple, real roots and these interlace,
2. $\delta_i'(\omega_o)\delta_r(\omega_o) - \delta_i(\omega_o)\delta_r'(\omega_o) > 0, \text{ for some } \omega_o \in (-\infty, \infty).$

Example

- Plant $$G(s) = \frac{1}{2s+1}, \quad C(s) = k_p + \frac{k_i}{s} = \frac{k_p s + k_i}{s}$$

- With k_p=1.8, k_i=0.2, we have $\delta(s) = 2s^2 + 2.8s + 0.2$ and it is stable.

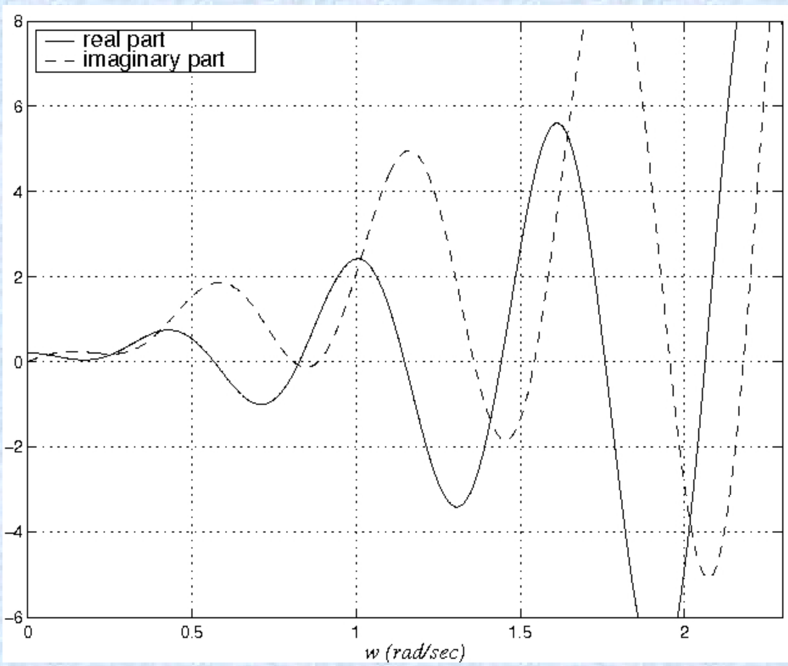
- Consider $$G(s) = \left[\frac{1}{2s+1}\right] e^{-10s}$$

- With k_p=1.8 and k_i=0.2, the characteristic equation of the closed-loop system is:
$$\delta(s) = 2s^2 + s + (1.8s + 0.2)e^{-10s} = 0$$

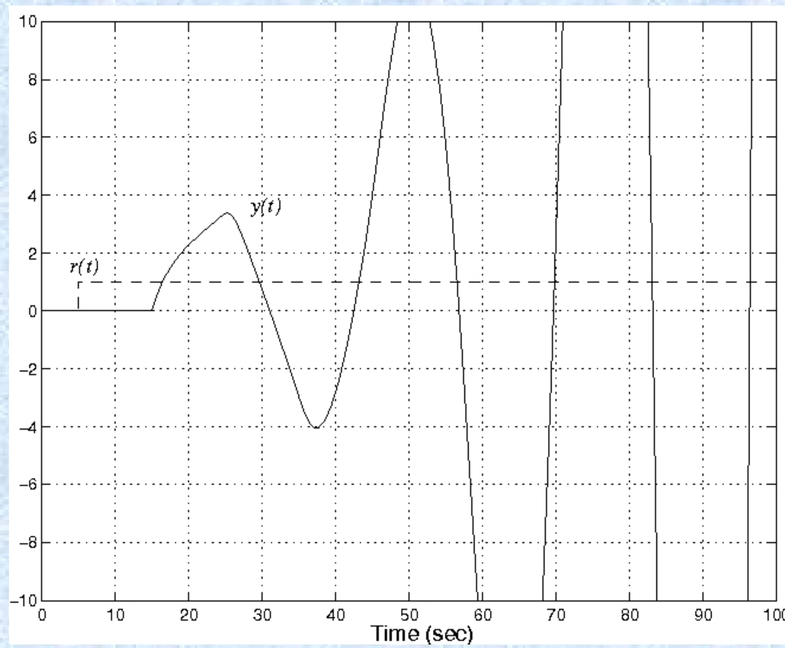
- For analyzing the stability, consider
$$\delta^*(s) = e^{10s}\delta(s) = (2s^2 + s)e^{10s} + 1.8s + 0.2$$

- The real and imaginary parts are given by

$$\begin{aligned}\delta_r(\omega) &= 0.2 - \omega\sin(10\omega) - 2\omega^2\cos(10\omega)\\ \delta_i(\omega) &= \omega[1.8 + \cos(10\omega) - 2\omega\sin(10\omega)]\ .\end{aligned}$$



Shows interlacing.



Shows instability

Analysis

1. The example illustrates the case of a time-delay system that satisfies the interlacing and monotonic phase increase properties but fails to be stable.
2. The reason for this behavior lies in the nature of the roots of real and imaginary parts of the polynomial: they are not all real.

THEOREM (Pontyagin)

Let M and N denote the highest powers of s and e^s, respectively, in $\delta^*(s)$. Let η be an appropriate constant such that the coefficients of terms of highest degree in $\delta_r(\omega)$ and $\delta_i(\omega)$ do not vanish at $\omega=\eta$. Then for the equations $\delta_r(\omega)=0$ or $\delta_i(\omega)=0$ to have only real roots, it is necessary and sufficient that in each of the intervals

$$-2l\pi + \eta \leq \omega \leq 2l\pi + \eta \qquad l = l_o, l_o+1, l_o+2, \cdots$$

$\delta_r(\omega)$ or $\delta_i(\omega)$ have exactly $4lN + M$ real roots for a sufficiently large l_0.

- Let $\hat{s} = 10s$

$$\hat{\delta}^*(\hat{s}) = (0.02\hat{s}^2 + 0.1\hat{s})e^{\hat{s}} + 0.18\hat{s} + 0.2$$

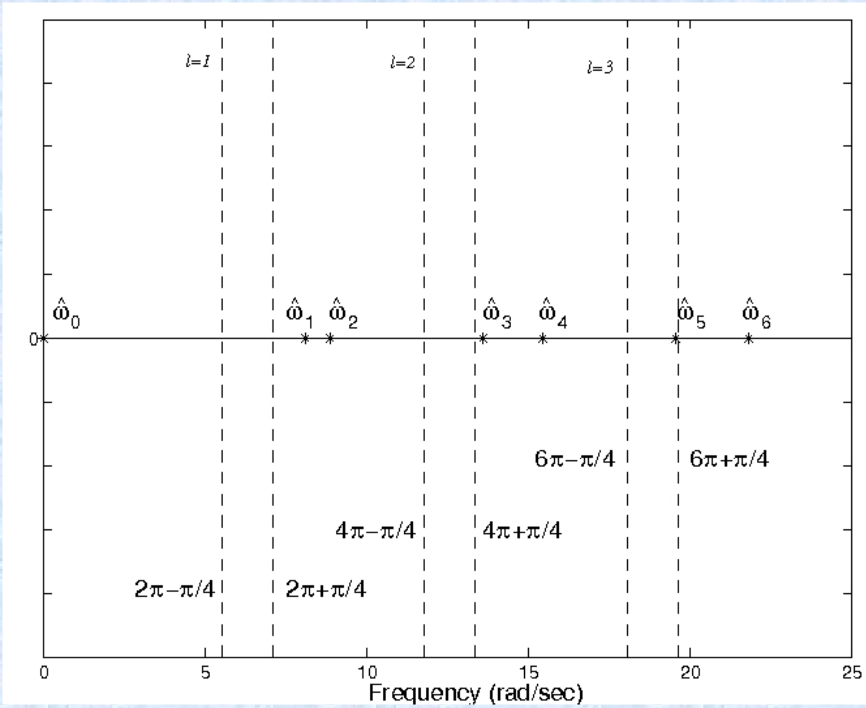
- The real and imaginary parts of the new quasi-polynomial is

$$\begin{aligned}\hat{\delta}_r(\hat{\omega}) &= 0.2 - 0.1\hat{\omega}\sin(\hat{\omega}) - 0.02\hat{\omega}^2\cos(\hat{\omega}) \\ \hat{\delta}_i(\hat{\omega}) &= \hat{\omega}[0.18 + 0.1\cos(\hat{\omega}) - 0.02\hat{\omega}\sin(\hat{\omega})] \ .\end{aligned}$$

- The roots of $\hat{\delta}_i(\hat{\omega}) = 0$

$$\hat{\omega}_o = 0; \quad \hat{\omega}_1 = 8.0812; \quad \hat{\omega}_2 = 8.8519; \quad \hat{\omega}_3 = 13.5896; \quad \hat{\omega}_4 = 15.4332;$$
$$\hat{\omega}_5 = 19.5618; \quad \hat{\omega}_6 = 21.8025; \quad \cdots$$

- Choose $\eta = \dfrac{\pi}{4}$



1. $\hat{\delta}_i(\hat{\omega})$ has only one real root in [0, 2π-π/4]; the root at the origin.
2. Since $\hat{\delta}_i(\hat{\omega})$ is an odd function, in the interval [-7π/4, 7π/4] , $\hat{\delta}_i(\hat{\omega})$ will have only one real root.
3. $\hat{\delta}_i(\hat{\omega})$ has no real roots in the interval [7π/4, 9π/4]; $\hat{\delta}_i(\hat{\omega})$ has only one real root in [-2π+π/4, 2π+π/4] which does not sum up to

$$4N + M = 6 \text{ for } l_0 = 1.$$

4. Let $l_0 = 2$ so the requirement on the number if real roots is $8N+M=10$. $\hat{\delta}_i(\hat{\omega})$ has only five real roots in [-4π+π/4, 4π+π/4].
5. Following the same procedure for $l = 3, 4, ...$ we see that the number of real roots of $\hat{\delta}_i(\hat{\omega})$ in [-2 lπ+π/4, 2 πl+π/4] is always less than $4lN + M = 4l + 2.$
6. We conclude that the roots of $\hat{\delta}_i(\hat{\omega})$ are not all real.

STABILITY OF SYSTEMS WITH A SINGLE DELAY

- Consider the characteristic equation

$$\delta(s, L) = d(s) + n(s)e^{-Ls} = 0$$

- Problem: Determine the ranges of values of *L* for which all the roots of the characteristic equation lie in the LHP.

- A systematic procedure to analyze the behavior of the roots of the characteristic polynomial as *L* increases from 0 to ∞.

Walton and Marshall's Procedure

Step 1: Examine the stability at $L=0$.

Step 2:

- Examine the behavior of the roots as increasing L from 0 to an infinitesimally small and positive.
- The number of roots changes from being finite to infinite. For an infinitesimally small L, the new roots must come in at infinity. Otherwise, $e^{-Ls} \approx 1$ and no new roots.
- Determine where in complex plane these new roots arise.
- If deg[n(s)]<deg[d(s)], the roots "s" is large iff e^{-Ls} is large (i.e., Re[s]<0) ⟹ New roots occur in the open LHP
- If deg[n(s)]=deg[d(s)], the location of the roots is determined by the sign of $W(\omega^2)$ for large ω.

Step 3:

- Examine potential crossing points on the imaginary axis (we separately consider the case s=0)
- Consider

$$\begin{cases} d(j\omega) + n(j\omega)e^{-jL\omega} & = 0 \\ d(-j\omega) + n(-j\omega)e^{jL\omega} & = 0 \end{cases} \Longrightarrow d(j\omega)d(-j\omega) - n(j\omega)n(-j\omega) = 0$$

$$\boxed{W(\omega^2) := d(j\omega)d(-j\omega) - n(j\omega)n(-j\omega)}$$

- If no positive roots of $W(\omega^2)$=0, then no values of L for which $\delta(j\omega, L) = 0$

Remark

If deg[n(s)]<deg[d(s)] and $W(\omega^2)$ has no positive real roots, then there is no change in stability:
The system will be stable for all $L \geq 0$ if the system is stable at L=0.
The system will be unstable for all $L \geq 0$ if the system is unstable at L=0.

Case when s=0

In this case, we have only one equation

$$d(0) + n(0) = 0 \quad \Rightarrow \quad d(0) + e^{-L0}n(0) = 0, \quad \text{for all finite } L$$

The system is unstable for all values of *L* and for analysis this solution can be ignored.

To find *L*,

$$\boxed{d(j\omega) + n(j\omega)e^{-jL\omega} = 0} \quad \Rightarrow \quad \boxed{e^{-jL\omega} = -\frac{d(j\omega)}{n(j\omega)}} \boxed{:= \cos(L\omega) - j\sin(L\omega)}$$

Once we have found a value of *L* at which there is a root of the characteristic equation on the imaginary axis, we need to determine if the root crosses the imaginary axis and in which direction or if it merely touches the imaginary axis.

$$\mathrm{Re}\left[\frac{ds}{dL}\right] > 0$$

destabilizing

$$\mathrm{Re}\left[\frac{ds}{dL}\right] < 0$$

stabilizing

$$\mathrm{Re}\left[\frac{ds}{dL}\right] = 0$$

Necessary to consider high-order derivatives

After some manipulations, we have

$$\boxed{S = \mathrm{sgn}\left[W'(\omega^2)\right] = \begin{cases} -1, & \text{destabilizing} \\ +1, & \text{stabilizing} \end{cases}}$$

Example

$$\delta(s, L) = s + 2e^{-Ls}$$

1. Examine $\delta(s,0)=s+2$, so the system is stable for $L=0$.
2. Since deg[d(s)]=1>deg[n(s)]=0, we skip step 2.
3. From d(s)=s, n(s)=2, we have $W(\omega^2)= \omega^2-4$.
   - $W'(\omega^2)=1>0$.
   - Since $S=\mathrm{sgn}[W'(\omega^2)]=1$, the root is destabilizing.
   - The corresponding values of L are

$$\begin{cases} \cos(L\omega) = \mathrm{Re}\left[-\dfrac{j\omega}{2}\right] = 0 \\ \sin(L\omega) = \mathrm{Im}\left[-\dfrac{j\omega}{2}\right] = 1 \end{cases} \Rightarrow L = (4k+1)\frac{\pi}{4}, k = 0, 1, 2, \cdots$$

- At $L=\pi/4$, two roots of $\delta(s,L)=0$ cross from left to right of the imaginary axis.
- At $L=5\pi/4$, two more roots cross from left to right of the imaginary axis and so on.

Conclusion

The region of stability is $0 \leq L < \pi/4$

FIRST ORDER SYSTEMS WITH TIME-DELAY

Plant: $$G(s) = \left[\frac{k}{1+Ts}\right] e^{-Ls}$$

PID Controller: $$C(s) = k_p + \frac{k_i}{s} + k_d s$$

Stability Conditions for Delay free Systems

Characteristic Polynomial without time-delay:

$$\delta(s) = (T + kk_d)s^2 + (1 + kk_p)s + kk_i$$

Assuming k>0, we have

$$\left\{k_p > -\frac{1}{k},\ k_i > 0,\ k_d > -\frac{T}{k}\right\} \quad \text{or} \quad \left\{k_p < -\frac{1}{k},\ k_i < 0,\ k_d < -\frac{T}{k}\right\}$$

Characteristic Polynomial with time-delay:

$$\delta(s) = (kk_i + kk_p s + kk_d s^2)e^{-Ls} + (1+Ts)s$$

Write

$$e^{Ls}\delta(s) = kk_i + kk_p s + kk_d s^2 + (1+Ts)se^{Ls} =: \delta^*(s)$$

Substituting s=jω,

$$\delta^*(j\omega) = \delta_r(\omega) + j\delta_i(\omega)$$

where

$$\begin{aligned}
\delta_r(\omega) &= kk_i - kk_d\omega^2 - \omega\sin(L\omega) - T\omega^2\cos(L\omega) \\
\delta_i(\omega) &= \omega\left[kk_p + \cos(L\omega) - T\omega\sin(L\omega)\right]
\end{aligned}$$

We now separately treat the two cases: open-loop stable and open-loop unstable plants.

Open-loop Stable Plant

Plant: $G(s) = \left[\dfrac{k}{1+Ts}\right] e^{-Ls}$ T>0 (for stable plants)

$$\delta^*(j\omega) = \delta_r(\omega) + j\delta_i(\omega)$$

$$\begin{aligned} \delta_r(\omega) &= kk_i - kk_d\omega^2 - \omega\sin(L\omega) - T\omega^2\cos(L\omega) \\ \delta_i(\omega) &= \omega\left[kk_p + \cos(L\omega) - T\omega\sin(L\omega)\right] \end{aligned}$$

- k_p only affects $\delta_i(\omega)$.
- k_i and k_d affect $\delta_r(\omega)$.
- Parameters appear affinely in $\delta_r(\omega)$ and $\delta_i(\omega)$.

For stability, $\delta_r(\omega)$ and $\delta_i(\omega)$ must have all real roots and these roots must interlace.

Lemma

The imaginary part of $\delta^*(j\omega)$ has only simple real roots iff

$$-\frac{1}{k} < k_p < \frac{1}{k}\left[\frac{T}{L}\alpha_1 \sin(\alpha_1) - \cos(\alpha_1)\right]$$

where α_1 is the solution of the equation

$$\tan(\alpha) = -\frac{T}{T+L}\alpha$$

in the interval $(0, \pi)$.

This lemma gives the ranges of k_p.

Let z=$\omega L \neq 0$, then

$$\delta_r(z) = \frac{k}{L^2} z^2 \left[-k_d + m(z)k_i + b(z)\right]$$

where

$$m(z) = \frac{L^2}{z^2}, \qquad b(z) = -\frac{L}{kz}\left[\sin(z) + \frac{T}{L} z \cos(z)\right]$$

Lemma

For each value of kp in the range, the necessary and sufficient conditions on k_i and k_d for the roots of $\delta_r(z)$ and $\delta_i(z)$ to interface is the following infinite set of inequalities:

$$k_i > 0, \quad k_d > m_1 k_i + b_1, \quad k_d < m_2 k_i + b_2, \quad k_d > m_3 k_i + b_3,$$
$$k_d < m_4 k_i + b_4, \quad \cdots$$

where the parameters m_j and b_j for j=1,2,3,… are given by

$$m_j := m(z_j), \qquad b_j := b(z_j).$$

Theorem

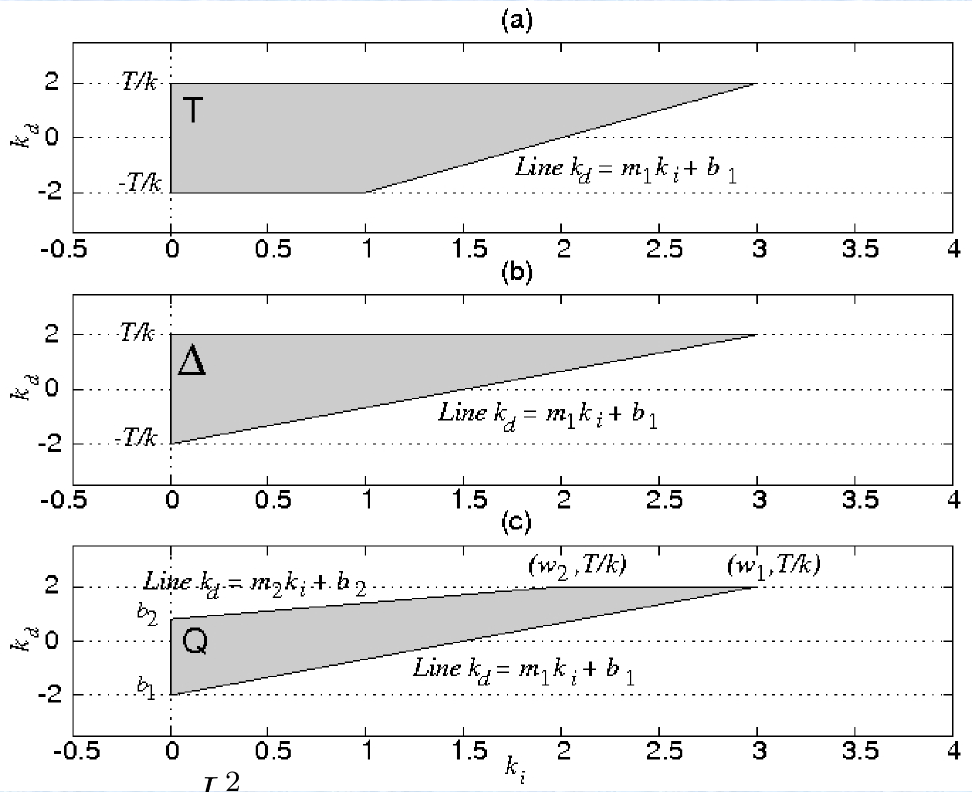
The range of k_p values for which a given open-loop stable plant, with transfer function considered, can be stabilized using a PID controller is given by

$$-\frac{1}{k} < k_p < \frac{1}{k}\left[\frac{T}{L}\alpha_1 \sin(\alpha_1) - \cos(\alpha_1)\right]$$

where α_1 is the solution of the equation

$$\tan(\alpha) = -\frac{T}{T+L}\alpha$$

in the interval $(0,\pi)$. For k_p values outside this range, there are no stabilizing PID controllers. The complete stabilizing region is given by:



For each $k_p \in (-1/k, 1/k)$, the cross-section of the stabilizing region in the (k_i, k_d) space is the trapezoid T;

For $k_p = 1/k$, the cross-section of the stabilizing region in the (k_i, k_d) space is the triangle Δ;

For each $k_p \in (1/k, k_u)$, the cross-section of the stabilizing region in the (k_i, k_d) space is the quadrilateral Q.

$$
\begin{aligned}
m_j &= \frac{L^2}{z_j^2}, \\
b_j &= -\frac{L}{k z_j}\left[\sin(z_j) + \frac{T}{L} z_j \cos(z_j)\right] \\
\omega_j &= \frac{z_j}{kL}\left[\sin(z_j) + \frac{T}{L} z_j (\cos(z_j) + 1)\right]
\end{aligned}
$$

where z_j are the real, positive solutions of $kk_p + \cos(z) - \frac{T}{L} z \sin(z) = 0$

Algorithm for Determining Stabilizing PID Parameters

1. Initialize k_p=-1/k and step=(k_u+1/k)/(N+1) where N is the desir ed number of points and

$$k_u = \frac{1}{k}\left[\frac{T}{L}\alpha_1 \sin(\alpha_1) - \cos(\alpha_1)\right]$$

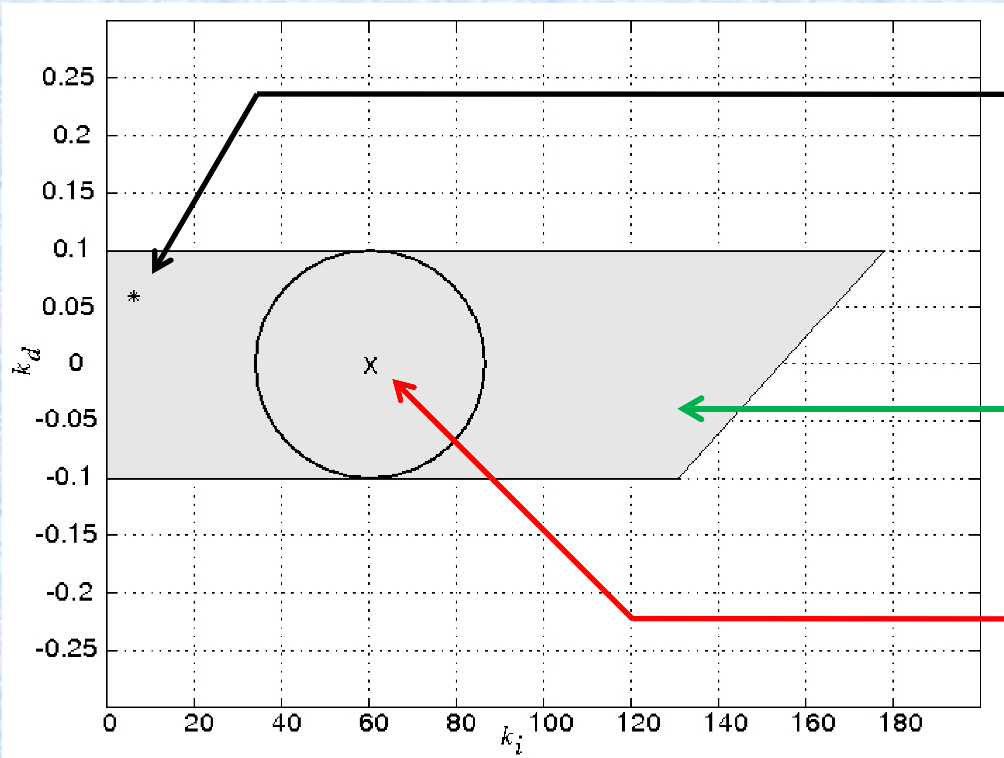
2. Set K_p=k_p+step;
3. If k_p<k_u, then go to 4. Else terminate the algorithm.
4. Find the roots z_1 and z_2 of

$$kk_p + \cos(z) - \frac{T}{L}z\sin(z) = 0.$$

5. Compute the parameters m_j and b_j, j=1,2 associated with the z_j.
6. Determine the stabilizing region in the (k_i,k_d) space.
7. Go to 2.

Example (Location of Z-N solution in the set)

$$G(s) = \left[\frac{0.1}{0.01s+1}\right] e^{-0.1s}$$



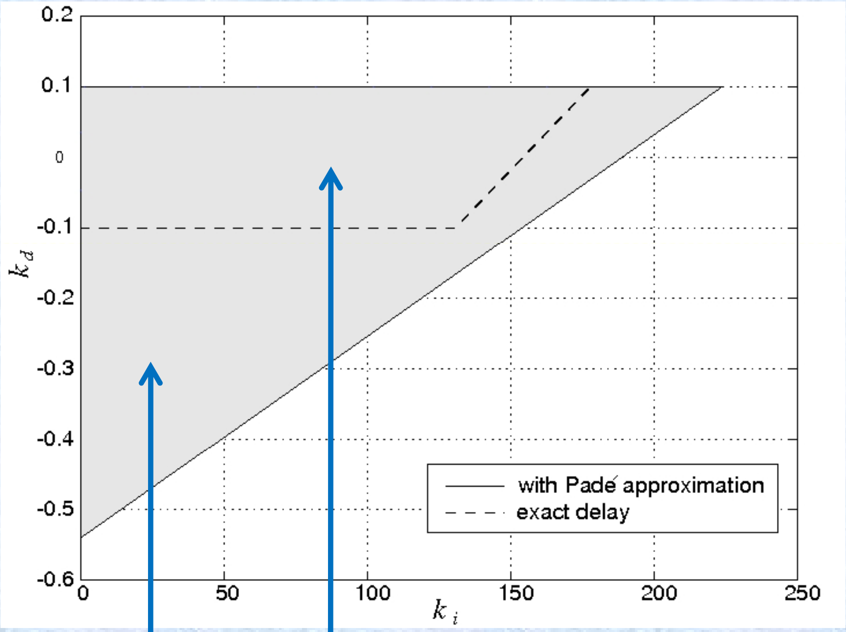
Stabilizing parameter set obtained by Ziegler-Nichols step response method.

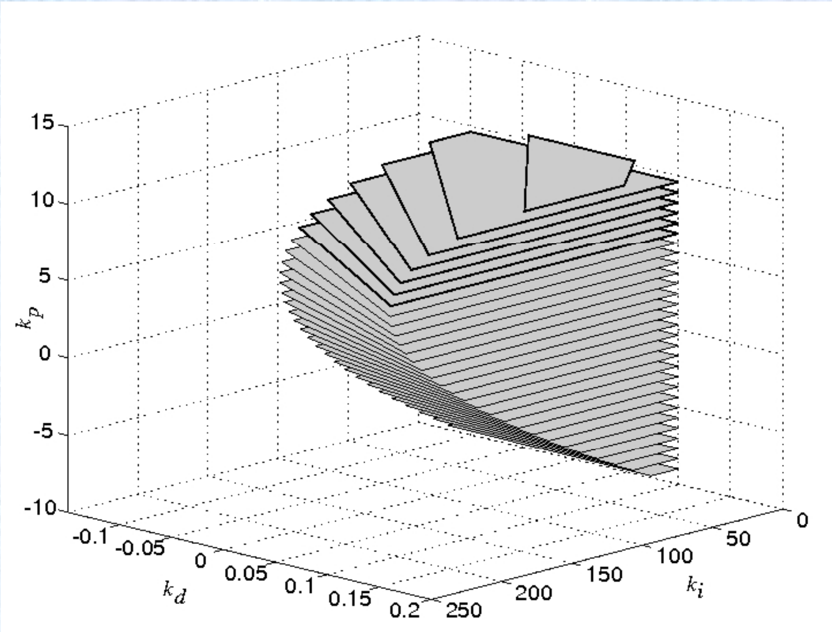
Stabilizing region

Stabilizing parameter set with the largest stability radius.

K_p=1.2

Example (Set using Pade Approximation vs. Set using a True Delay System)





3D stabilizing set

Set from the true delay system

Set from the 1st order Pade approximation
(It contains destabilizing parameters)

Open-loop Unstable Plant

Plant: $G(s) = \left[\frac{k}{1+Ts}\right] e^{-Ls}$ ⇒ T<0 (for unstable plants)

Lemma

For |T/L|>0.5, $\delta_i(j\omega)$ has only simple real roots iff

$$\frac{1}{k}\left[\frac{T}{L}\alpha_1 \sin(\alpha_1) - \cos(\alpha_1)\right] < k_p < -\frac{1}{k}$$

where α_1 is the solution of the equation

$$\tan(\alpha) = -\frac{T}{T+L}\alpha$$

in the interval (0,π). In the special case of |T/L|=1, we have $\alpha_1=\pi/2$.
For |T/L|≤0.5, the roots of $\delta_i(j\omega)$ are not all real.

Let $z=\omega L\neq 0$,

$$\delta_r(z) = \frac{k}{L^2} z^2 \left[-k_d + m(z)k_i + b(z)\right]$$

where $$m(z) = \frac{L^2}{z^2}, \qquad b(z) = -\frac{L}{kz}\left[\sin(z) + \frac{T}{L} z\cos(z)\right]$$

Lemma

For each value of k_p in the range, the necessary and sufficient conditions on k_i and k_d for the roots of $\delta_r(z)$ and $\delta_i(z)$ to interlace are the following infinite set of inequalities:

$$k_i < 0, \quad k_d < m_1 k_i + b_1, \quad k - d > m_2 k_i + b_2, \quad k_d < m_3 k_i + b_3,$$
$$k_d > m_4 k_i + b_4, \cdots$$

where the parameters m_j and b_j for j=1,2,3,… are given by

$$m_j := m(z_j), \qquad b_j := b(z_j)$$

Theorem

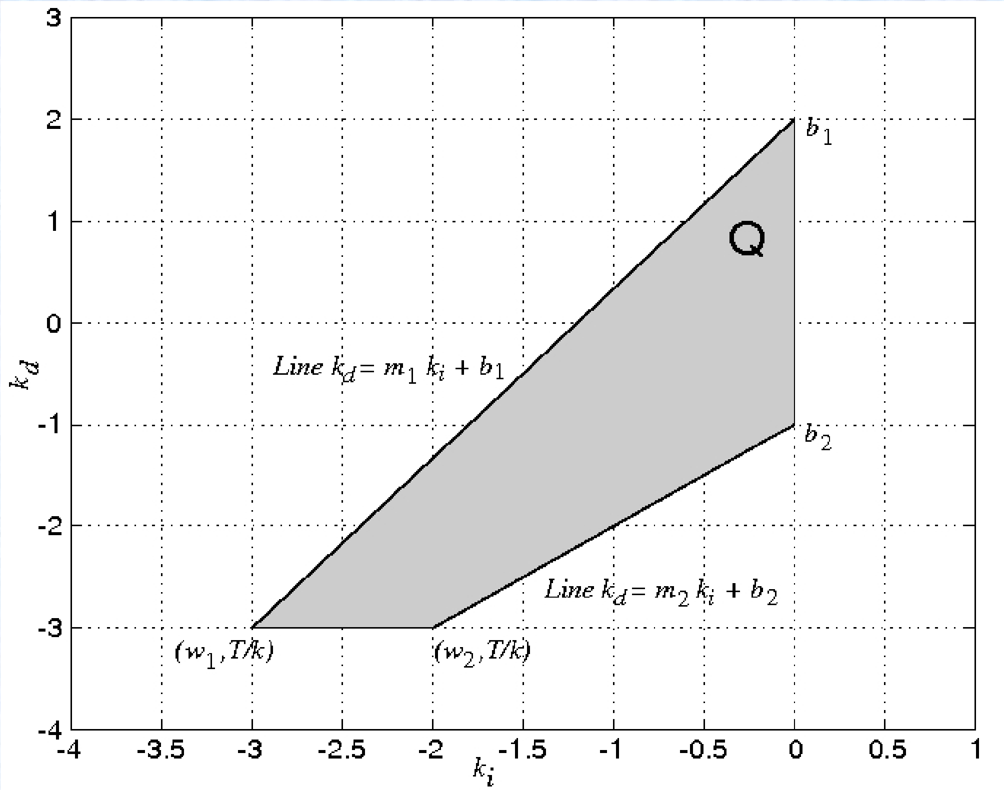
A necessary and sufficient condition for the existence of a stabilizing PID controller for the open-loop unstable plant considered is $|T/L|>0.5$. If this condition is satisfied, then the range of k_p values for which a given open-loop unstable plant, with transfer function considered, can be stabilized using a PID controller is given by

$$\frac{1}{k}\left[\frac{T}{L}\alpha_1 \sin(\alpha_1) - \cos(\alpha_1)\right] < k_p < -\frac{1}{k}$$

where α_1 is the solution of the equation

$$\tan(\alpha) = -\frac{T}{T+L}\alpha$$

in the interval $(0,\pi)$. In the special case of $|T/L|=1$, we have $\alpha_1=\pi/2$. For k_p values outside this range, there are no stabilizing PID controllers. Moreover, the complete stabilizing region is given:



For each $k_p \in (k_l, -1/k)$, the cross-section of the stabilizing region in the (k_i, k_d) space is quadrilateral Q.

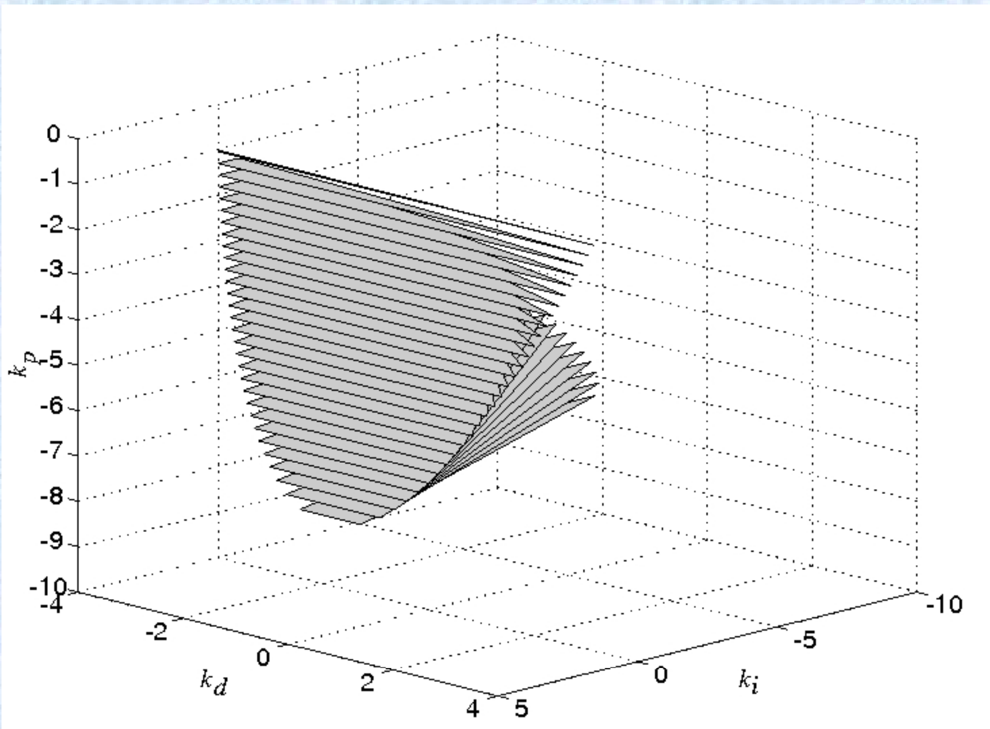
The stabilizing region of (k_i, k_d) for $k_l < k_p < -1/k$ where

$$k_l := \frac{1}{k}\left[\frac{T}{L}\alpha_1 \sin(\alpha_1) - \cos(\alpha_1)\right]$$

Example

Consider a process defined by

$$\frac{dy(t)}{dt} = 0.25y(t) - 0.25u(t-0.8) \quad \Rightarrow \quad G(s) = \frac{1}{1-4s}e^{-0.8s}$$



The stabilizing region of (k_p, k_i, k_d) values for the PID controllers. $(-8.6876 < k_p < -1)$

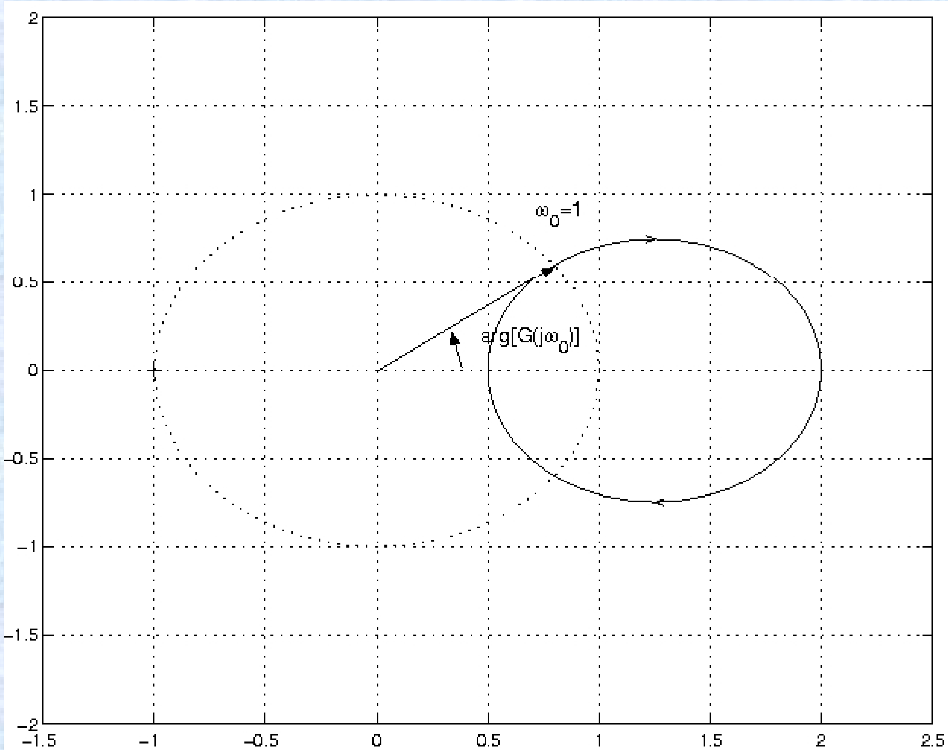
ARBITRARY LTI SYSTEMS WITH A SINGLE TIME-DELAY

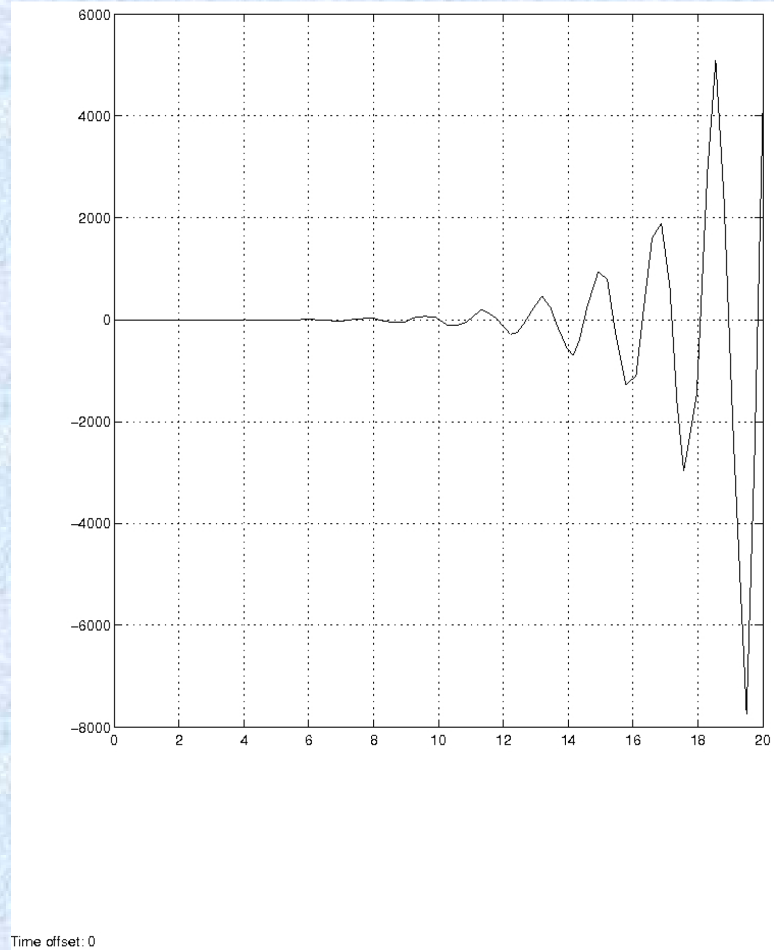
Tsypkin proposed a method to extend the Nyquist criterion to deal with time-delay systems (1946). This may lead to misleading conclusions unless care is taken.

Example

$$G(s) = \frac{2s+1}{s+2}$$

- The closed-loop system is stable with unity negative feedback.
- According to Tsypkin, the closed-loop system should tolerate a time-delay upto 3.7851.
- However, when we add a 1 second delay to the nominal transfer function, the closed-loop system becomes unstable.





- The Nyquist plot intersects the unit circle at ω_0=1.
- The closed-loop system should tolerate a time-delay upto

$$L_0 = \frac{\pi + \arg G(j\omega_0)}{\omega_0} = 3.7851.$$

The closed-loop system is unstable with a 1 second delay.

Pontryagin's Theory vs. the Nyquist Criterion

Let $h(z,t)$ be a polynomial in the two variable z and t with constant coefficients,

$$h(z,t) = \sum_{m,n} a_{mn} z^m t^n$$

The term $a_{rs} z^r t^s$ is called the principle term of the polynomial if $a_{rs} \neq 0$ and r and s each attain their maximum.

Write

$$h(z,t) = \chi_r^{(s)}(t) z^r + \chi_{r-1}^{(s)}(t) z^{r-1} + \cdots + \chi_1^{(s)}(t) z + \chi_0^{(s)}(t),$$

where $\chi_j^{(s)}(t), \quad j = 0, 1, 2, \ldots, r$ are polynomials in t with degree at most equal to s.

Two Theorems of Pontryagin to Clarify Nyquist Criterion Based Conditions for Systems with Time-delay

Theorem

If the polynomial $h(z,t) = \sum_{m,n} a_{mn} z^m t^n$ has no principal term, then the function

$$H(z) = h(z, e^z)$$

has an unbounded number of zeros with arbitrary large positive real part.

Theorem

Let $H(z)=h(z,e^z)$ where $h(z,t)$ is a polynomial with principal term $a_{rs}z^r t^s$. If the function $\chi_r^{(s)}(e^z)$ has roots in the open RHP, then the function $H(z)$ has an unbounded set of zeros in the open RHP. If all the zeros of the function $\chi_r^{(s)}(e^z)$ lie in the open LHP, then the function $H(z)$ has no more than a bounded set of zeros in the open RHP.

Conditions which should be satisfied when using the Nyquist criterion with the conventional Nyquist contour

Theorem

Suppose that we are given a unity feedback system with an open-loop transfer function

$$G(s) = G_0(s)e^{-Ls} = \left[\frac{N(s)}{D(s)}\right] e^{-Ls}$$

where $N(s)$ and $D(s)$ are real polynomials of degree m and n, respectively and L is a fixed delay. Then we have the following conclusions:

1. If $n<m$, or, $n=m$ and $|b_n/a_n|\geq 1$ where an, bn are the leading coefficients of $D(s)$ and $N(s)$, respectively, the Nyquist criterion is not applicable and the system is unstable according to Pontryagin's theorems.

2. If $n>m$, or, $n=m$ and $|b_n/a_n|<1$, the Nyquist criterion is applicable and we can use it to check the stability of the closed-loop system.

It is appropriate to point out that most likely Typkin assumed the plant to be strictly proper, though he did not state it explicitly in the literature. Attaching a PID controller to a proper or strictly proper plant opens up the very real possibility of ending up with an improper or a proper open-loop transfer function. This is the reason that the above investigation had to be undertaken.

Solution Approach

1. Find the complete set of $\mathbf{k}$'s which stabilize the delay-free plant $P_0(s)$ and denote this set as S_0.

2. Define the set S_N, which is the set of $\mathbf{k}$'s such that $C(s,\mathbf{k})P_0(s)$ is an improper transfer function or

$$\lim_{s\to\infty} |C(s,\mathbf{k})P_0(s)| \geq 1$$

   Note that the elements in S_N make the closed-loop system unstable after the delay is introduced. Exclude S_N from S_0 and denote the new set by S_1, that is, $\mathcal{S}_1 = \mathcal{S}_0 \backslash \mathcal{S}_N$

3. Compute the set S_L:

$$\mathcal{S}_L = \left\{\mathbf{k} \mid \mathbf{k} \notin \mathcal{S}_N \text{ and } \exists L \in [0, L_0], \omega \in \mathbf{R}, \text{s.t.} C(j\omega)P_0(j\omega)e^{-jL\omega} = -1\right\}$$

   S_L is the set of $\mathbf{k}$'s which make $C(s,\mathbf{k})P(s)$ have a minimal destabilizing delay that is less than or equal to L_0.

4. The set $\mathcal{S}_R = \mathcal{S}_1 \backslash \mathcal{S}_L$ is the solution

Theorem

The set of controllers $C(s,\mathbf{k})$ denoted by S_R is the complete set of controllers in the unity feedback configuration that stabilize the plant $P(s)$ with delay L from 0 up to L_0.

Proportional Controllers

Plant and controller: $$P(s) = P_0(s)e^{-Ls} = \left[\frac{N(s)}{D(s)}\right]e^{-Ls}, \qquad C(s) = k_p$$

To implement the method, the key is to find S_L.

The point the Nyquist curve crossing (-1,0): Find *L* and *ω* satisfying

$$\boxed{C(j\omega)P_0(j\omega)e^{-jL\omega} = -1}$$

$$\Downarrow$$

$$\begin{aligned} \arg[k_p P_0(j\omega)] - L\omega &= 2h\pi - \pi, \quad h \in \mathbf{Z} \\ |k_p P_0(j\omega)| &= 1. \end{aligned}$$

$$\Downarrow$$

$$\begin{aligned} L(\omega, k_p) &= \frac{\arg[k_p P_0(j\omega)] + \pi}{\omega} \\ k_p(\omega) &= \pm\frac{1}{|P_0(j\omega)|}. \end{aligned}$$

- For $k_p>0$,

$$L(\omega, k_p) = L(\omega) = \frac{\arg[P_0(j\omega)] + \pi}{\omega}$$

Solve $L(\omega) \le L_0$ to get a set of ω: Ω^+
Set of $k_p>0$ corresponding to Ω^+: S_L^+

S_L^+ consists of all the positive k_p's that make the system have poles on the imaginary axis for certain $L \le L_0$.

- For $K_p<0$,
Ω^- : a set of ω for $L(\omega) \le L_0$
S_L^- : a set of $k_p<0$ corresponding to Ω^-

The complete set S_L: $\mathcal{S}_L = \mathcal{S}_L^+ \cup \mathcal{S}_L^-$

Algorithm for P Controllers

1. Compute the delay-free stabilizing k_p set, S_0
2. Find S_N
   - If deg[$N(s)$]>deg[$D(s)$], S_N=**R**. i.e., S_R=∅
   - If deg[$N(s)$]<deg[$D(s)$], S_N= ∅
   - If deg[$N(s)$]=deg[$D(s)$],

$$\mathcal{S}_N = \left\{ k_p \mid |k_p| \geq \left| \frac{a_n}{b_n} \right| \right\},$$

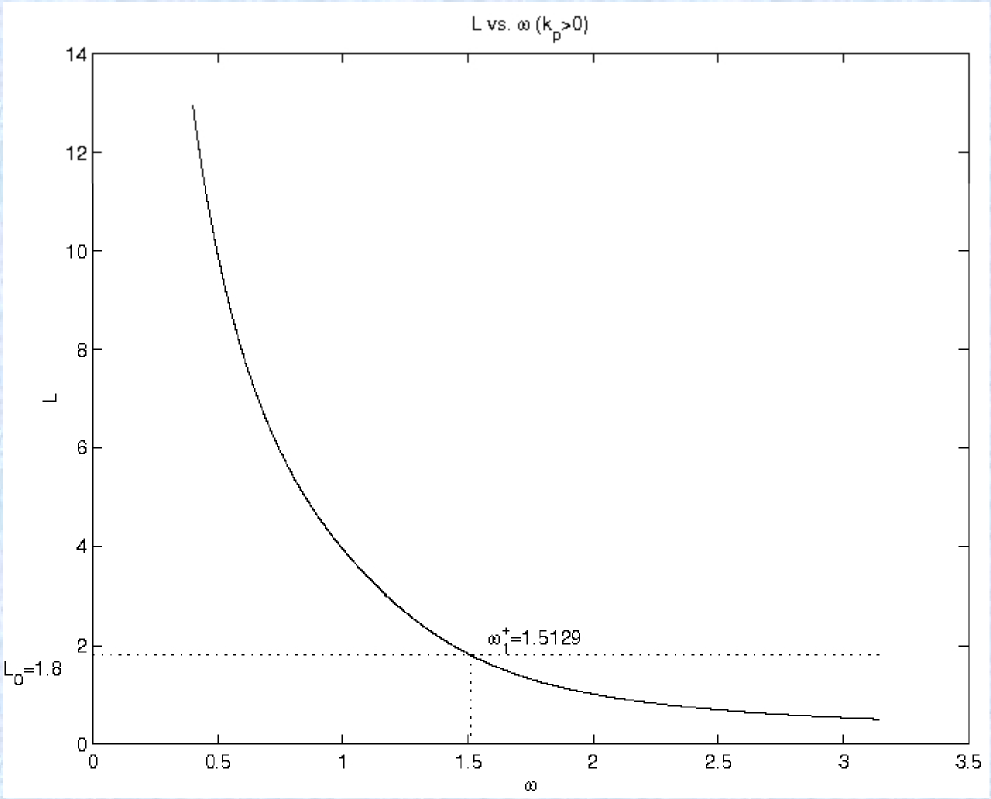
   where an, b_n are the leading coefficients of $D(s)$ and $N(s)$.

3. Compute $\mathcal{S}_1 = \mathcal{S}_0 \backslash \mathcal{S}_N$
4. Compute S_L
5. Compute $\mathcal{S}_R = \mathcal{S}_1 \backslash \mathcal{S}_L$

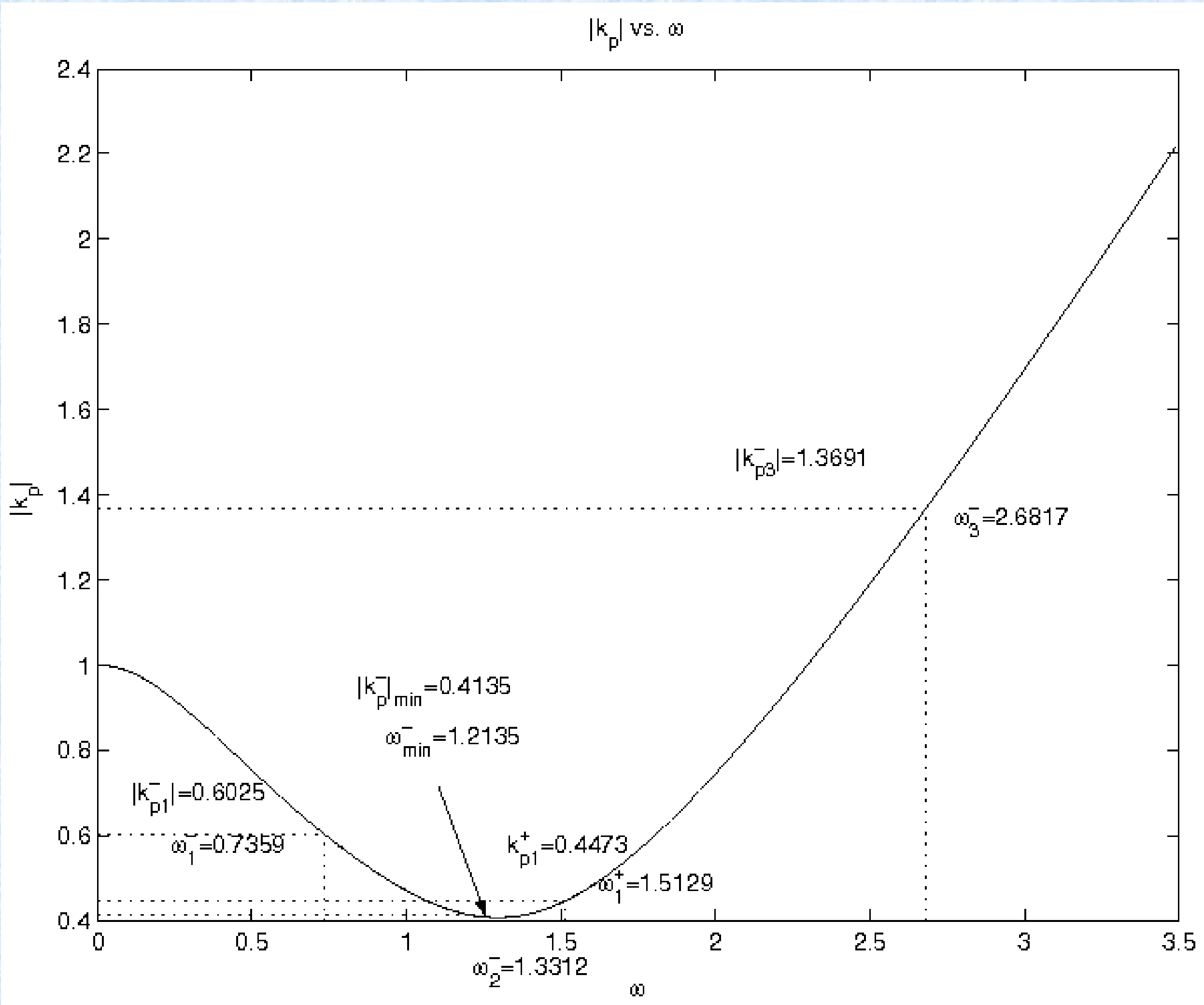
Example

$$P(s) = \left[\frac{s^2+3s-2}{s^3+2s^2+3s+2}\right] e^{-Ls} \text{ with delay up to } L_0 = 1.8$$

- For the delay-free plant, the stabilizing k_p range S_0=(-0.4093,1).
- Since deg[$N(s)$]=2<deg[$D(s)$], S_N=∅ and S_1=S_0
- For k_p>0, Ω^+ =[1.5129, +∞)

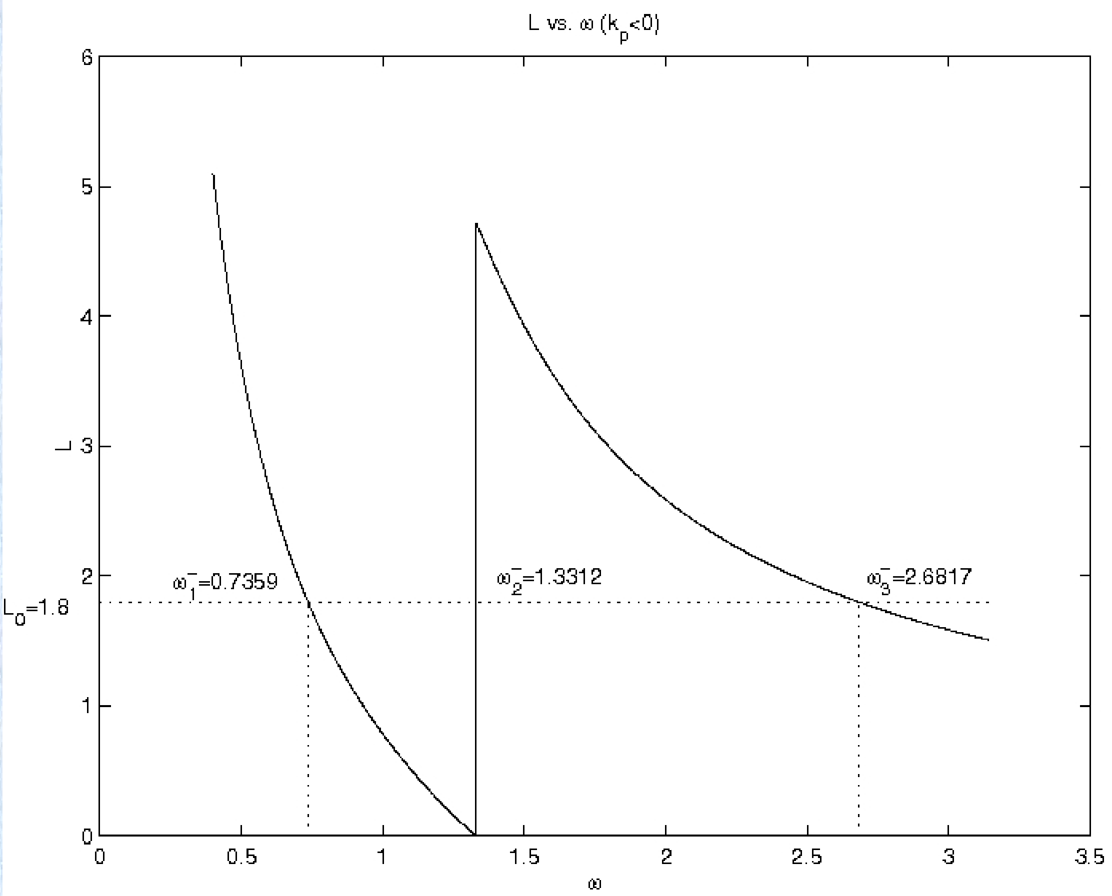


- The corresponding $S_{L^+} = [0.4473, +\infty)$



For $k_p<0$, $\Omega^- = [0.7359, 1.3312] \cup [2.6817, +\infty]$

The corresponding $S_L^- = [-0.6025, -0.4135] \cup [-\infty, -1.3691]$



$$\begin{aligned}\mathcal{S}_R &= \mathcal{S}_1 \backslash \mathcal{S}_L \\ &= (-0.4093, 1) \backslash ([0.4473, +\infty) \cup [-0.6025, -0.4135] \cup (-\infty, -1.3691]) \\ &= (-0.4093, 0.4473)\end{aligned}$$

PI Controllers

PI Controller: $$C(s) = k_p + \frac{k_i}{s} = \frac{k_p s + k_i}{s}$$

Open-loop transfer function:

$$G(s) = C(s)P(s) = C(s)P_0(s)e^{-Ls} = G_0(s)e^{-Ls}$$

Consider
$$\begin{aligned} G_0(s) = C(s)P_0(s) &= \left(\frac{k_p s + k_i}{s}\right)\frac{N(s)}{D(s)} \\ &= (k_p s + k_i)\underbrace{\frac{N(s)}{sD(s)}}_{R_0(s)} \end{aligned}$$

Magnitude and phase conditions

$$\begin{aligned} \arg[(k_i + jk_p\omega)R_0(j\omega)] - L\omega &= -\pi \\ |(k_i + jk_p\omega)R_0(j\omega)| &= 1 \end{aligned}$$

Rewrite the magnitude and phase conditions,

$$\begin{aligned} L(\omega, k_p, k_i) &= \frac{\arg[(k_i + jk_p\omega)R_0(j\omega)] + \pi}{\omega} \\ k_i &= \pm\sqrt{\frac{1}{|R_0(j\omega)|^2} - k_p^2\omega^2}. \end{aligned}$$

Fix k_p, then

$$M(\omega) = \frac{1}{|R_0(j\omega)|^2} - k_p^2\omega^2 \quad \Longrightarrow \quad k_i = \pm\sqrt{M(\omega)}$$

Note that only those ω's with $M(\omega) \geq 0$ need consideration when computing S_L.

Algorithm for PI Controllers

1. Compute S_0
2. Compute S_N
   - If deg[$N(s)$]>deg[$D(s)$], S_N=$\mathbf{R}^2$, i.e., S_R=∅
   - If deg[$N(s)$]<deg[$D(s)$], S_N= ∅
   - If deg[$N(s)$]=deg[$D(s)$], $\mathcal{S}_N = \left\{(k_p, k_i) | k_p, k_i \in \mathbf{R} \text{ and } |k_p| \geq \left|\frac{a_n}{b_n}\right|\right\}$

     where a_n, b_n are leading coefficients of $D(s)$ and $N(s)$.
3. Compute $\mathcal{S}_1 = \mathcal{S}_0 \backslash \mathcal{S}_N$
4. For a fixed k_p, find $S_{R,kp}$
   - Determine the sets Ω^+ and $\mathcal{S}^+_{L,k_p}$:
   - Determine the sets Ω^- and $\mathcal{S}^-_{L,k_p}$:

5. Compute

$$\begin{aligned} \mathcal{S}_{L,k_p} &= \mathcal{S}^{+}_{L,k_p} \cup \mathcal{S}^{-}_{L,k_p} \\ \mathcal{S}_{R,k_p} &= S_{1,k_p} \backslash \mathcal{S}_{L,k_p} \end{aligned}$$

6. By sweeping over k_p, the complete set of PI controllers that stabilize all plant with delay up to L_0

$$\mathcal{S}_R = \bigcup_{k_p} \mathcal{S}_{R,k_p}$$

PID Controllers for an Arbitrary LTI Plant with Delay

$$G(s) = C(s)P_0(s)e^{-Ls} = G_0(s)e^{-Ls}$$

where

$$\begin{aligned} G_0(s) = C(s)P_0(s) &= \frac{k_d s^2 + k_p s + k_i}{s} \cdot \frac{N(s)}{D(s)} \\ &= (k_d s^2 + k_p s + k_i) \underbrace{\left[\frac{N(s)}{sD(s)}\right]}_{R_0(s)} \end{aligned}$$

The magnitude and phase conditions:

$$\begin{aligned} \arg[(k_i - k_d\omega^2 + jk_p\omega)R_0(j\omega)] - L\omega &= -\pi \\ |(k_i - k_d\omega^2 + jk_p\omega)R_0(j\omega)| &= 1 \end{aligned}$$

Rewrite the phase and magnitude conditions,

$$
\begin{aligned}
L(\omega, k_p, k_i, k_d) &= \frac{\pi + \arg\left([(k_i - k_d\omega^2) + jk_p\omega] \cdot R_0(j\omega)\right)}{\omega} \\
k_i - k_d\omega^2 &= \pm\sqrt{\frac{1}{|R_0(j\omega)|^2} - (k_p\omega)^2}.
\end{aligned}
$$

For fixed k_p,

$$M(\omega) = \frac{1}{|R_0(j\omega)|^2} - (k_p\omega)^2 \quad \Rightarrow \quad k_i - k_d\omega^2 = \pm\sqrt{M(\omega)}$$

Similar to the PI case, we only need to consider ω's with $M(\omega) \geq 0$ when computing S_L.

Algorithm for PID Controllers

1. Compute S_0
2. Compute S_N
   - If deg[$N(s)$]>deg[$D(s)$]-1, S_N=$\mathbf{R}^3$, i.e., S_R=∅
   - If deg[$N(s)$]<deg[$D(s)$]-1, S_N= ∅
   - If deg[$N(s)$]=deg[$D(s)$]-1,

$$\mathcal{S}_N = \left\{ (k_p, k_i, k_d) | k_p, k_i, k_d \in \mathbf{R} \ \text{ and } \ |k_d| \geq \left| \frac{a_n}{b_{n-1}} \right| \right\}$$

   where a_n, b_{n-1} are leading coefficients of $D(s)$ and $N(s)$.

3. Compute $\mathcal{S}_1 = \mathcal{S}_0 \backslash \mathcal{S}_N$
4. For a fixed k_p, determine the set $S_{R,kp}$

- Determine the set Ω^+ and $S_{L,kp}{}^+$

$$\Omega^+ = \Big\{\omega \mid \omega > 0 \text{ and } M(\omega) \geq 0 \text{ and}$$
$$L(\omega) = \frac{\pi + \arg\{[\sqrt{M(\omega)} + jk_p\omega] \cdot R_0(j\omega)\}}{\omega} \leq L_0\Big\}$$
$$\mathcal{S}^+_{L,k_p} = \Big\{(k_i, k_d) \mid (k_i, k_d) \notin \mathcal{S}_{N,k_p} \text{ and } \exists\, \omega \in \Omega^+$$
$$\text{such that } \; k_i - k_d\omega^2 = \sqrt{M(\omega)}\Big\}.$$

  Note that $S_{L,kp}{}^+$ is a set of straight lines in the (k_i,k_d) space.

- Determine the sets Ω^- and $S_{L,kp}{}^-$
- Compute $\mathcal{S}_{L,k_p} = \mathcal{S}^+_{L,k_p} \cup \mathcal{S}^-_{L,k_p}$ and $\mathcal{S}_{R,k_p} = \mathcal{S}_{1,k_p} \backslash \mathcal{S}_{L,k_p}$

5. By sweeping over k_p, the complete set of PID controllers that stabilize all plants with delay up to L_0:

$$\mathcal{S}_R = \bigcup_{k_p} \mathcal{S}_{R,k_p}$$

Example

$$P(s) = \frac{k}{Ts+1} e^{-Ls}, \quad L \in [0, L_0]$$

The stabilizing PID parameters for the delay-free plant are:

$$\mathcal{S}_0 = \left\{ (k_p, k_i, k_d) \mid k_p > -\frac{1}{k}, k_i > 0, k_d > -\frac{T}{k} \text{ or } k_p < -\frac{1}{k}, k_i < 0, k_d < -\frac{T}{k} \right\}$$

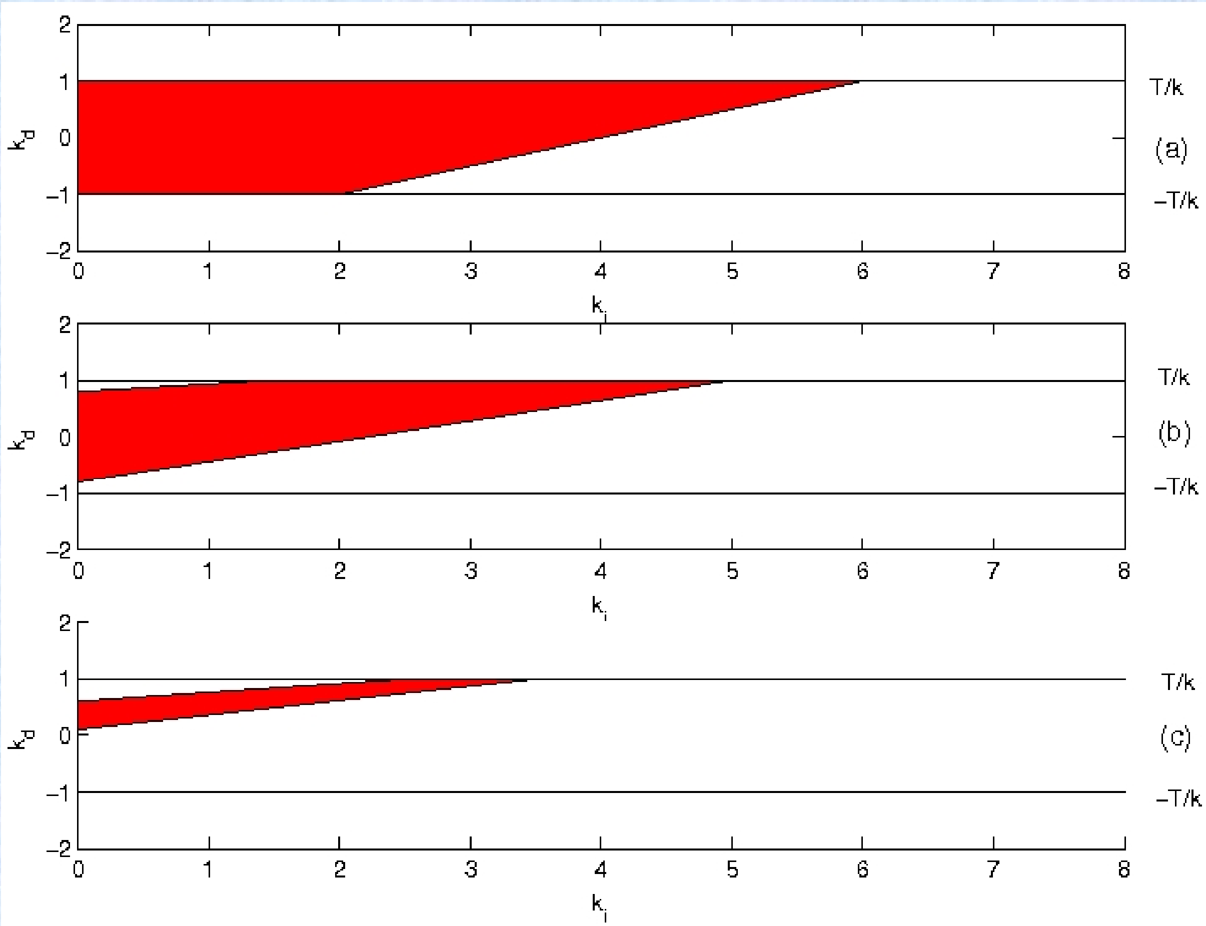
Since deg[$D(s)$]-deg[$N(s)$]=1,

$$\mathcal{S}_N = \left\{ (k_p, k_i, k_d) \mid k_p, k_i, k_d \in \mathbf{R} \text{ and } |k_d| \geq \left| \frac{T}{k} \right| \right\}$$

Assuming k>0, we have

$$\mathcal{S}_1 = \mathcal{S}_0 \backslash \mathcal{S}_N = \begin{cases} \left\{ (k_p, k_i, k_d) \mid k_p > -\frac{1}{k},\ k_i > 0,\ \frac{T}{k} > k_d > -\frac{T}{k} \right\} & \text{for } T > 0 \\ \left\{ (k_p, k_i, k_d) \mid k_p < -\frac{1}{k}, k_i < 0, \frac{T}{k} < k_d < -\frac{T}{k} \right\} & \text{for } T < 0 \end{cases}$$

For T>0, with different k_p values, the stabilizing regions of (k_i,k_d) take on different but simple shapes:



For $-1/k<k_p\leq 1/k$, $S_{R,kp}$ is a trapezoid. (a)

For $k_p>1/k$, $S_{R,kp}$ is a quadrilateral. (b) and (c)

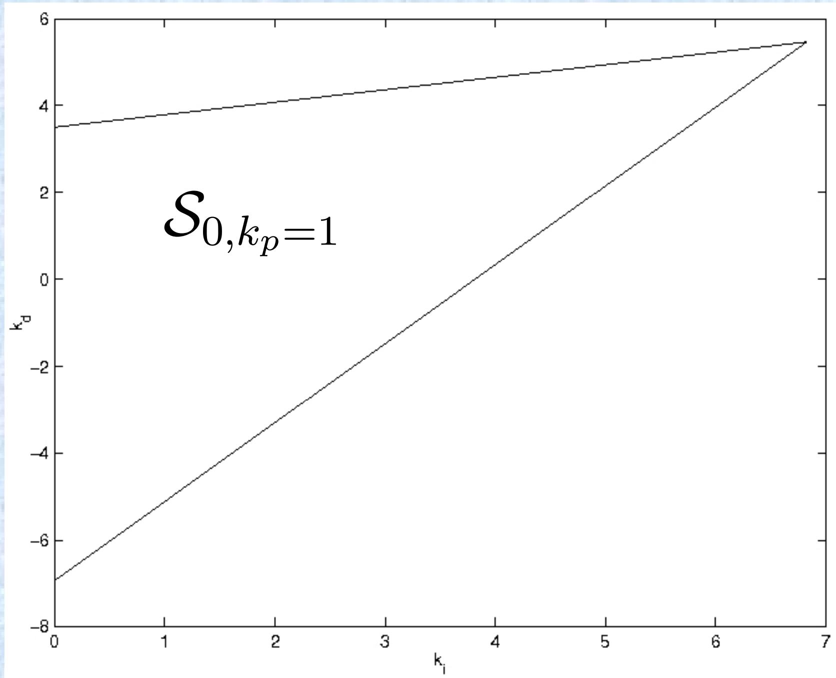
Example

$$P(s) = \left[\frac{s^3 - 4s^2 + s + 2}{s^5 + 8s^4 + 32s^3 + 46s^2 + 46s + 17} \right] e^{-Ls}$$

with L up to L_0=1, that is, for all $L \in [0, 1]$.

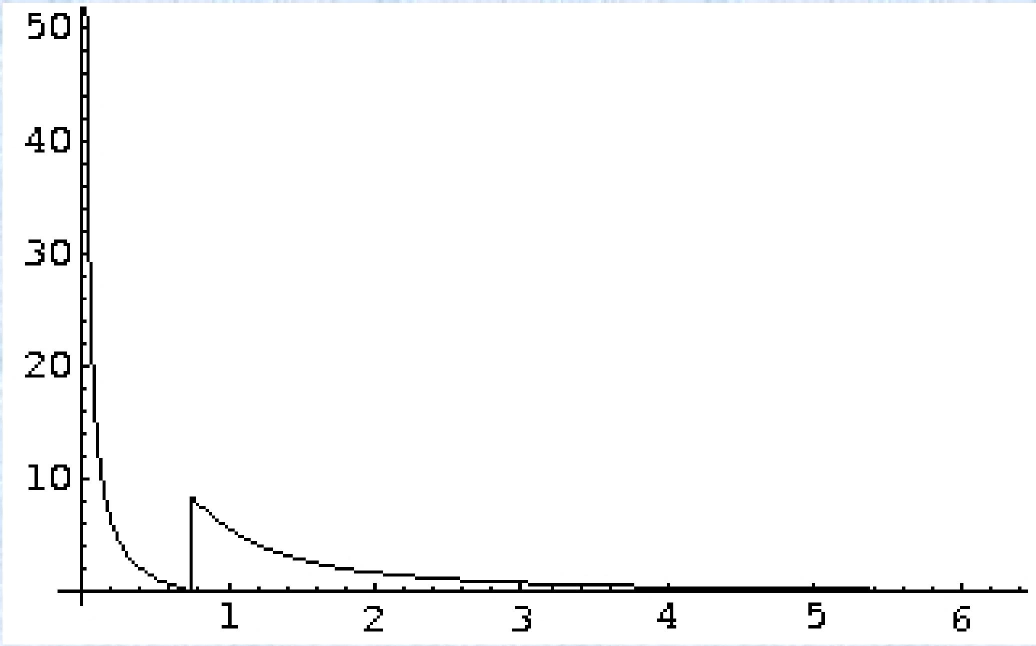
- Fix k_p=1, compute the stabilizing k_i, k_d values for the delay-free plant, say $S_{0,kp}$.

Stabilizing region of (k_i,k_d) with k_p=1 for delay-free system



- Since deg[$D(s)$]-deg[$N(s)$]>1, $S_N=\emptyset$ and $S_1=S_0$.
- For $k_i - k_d\omega^2 = \sqrt{M(\omega)} > 0,$ the set of ω where $L(\omega)\leq L_0$ is

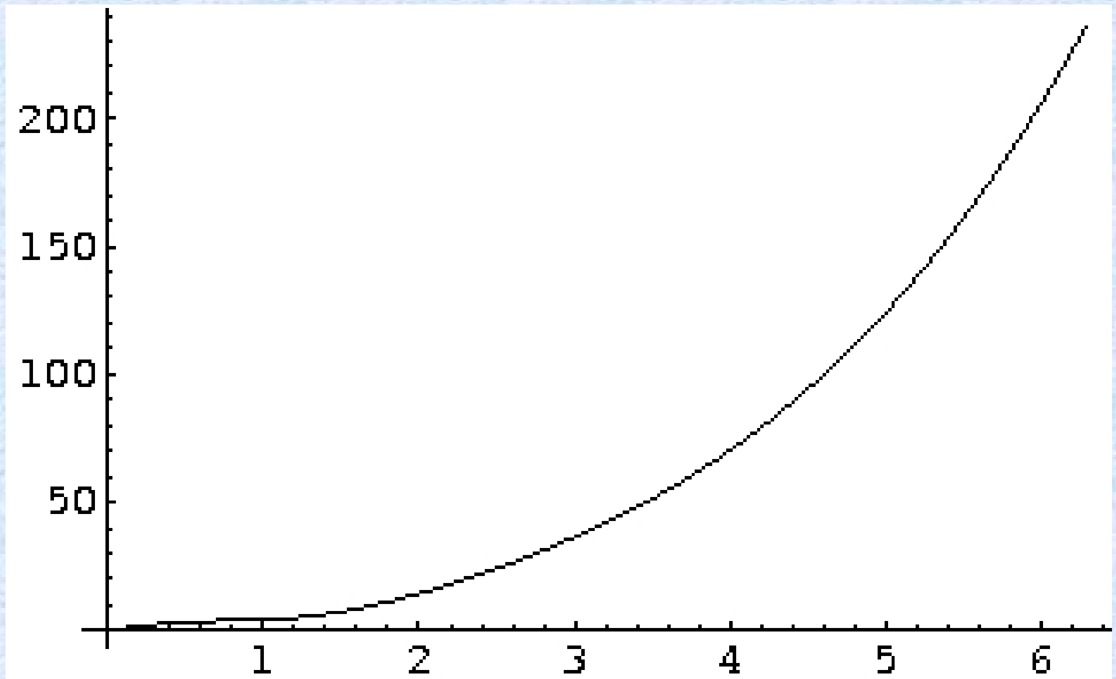
$$\Omega^+ = [0.524825, 0.742302] \cup [2.57318, +\infty)$$



$L(\omega)$ vs. ω with $k_i - k_d\omega^2 = \sqrt{M(\omega)}$

- The corresponding values of $\sqrt{M(\omega)}$
- $S_{L,kp}{}^{+}$: the straight lines defined by

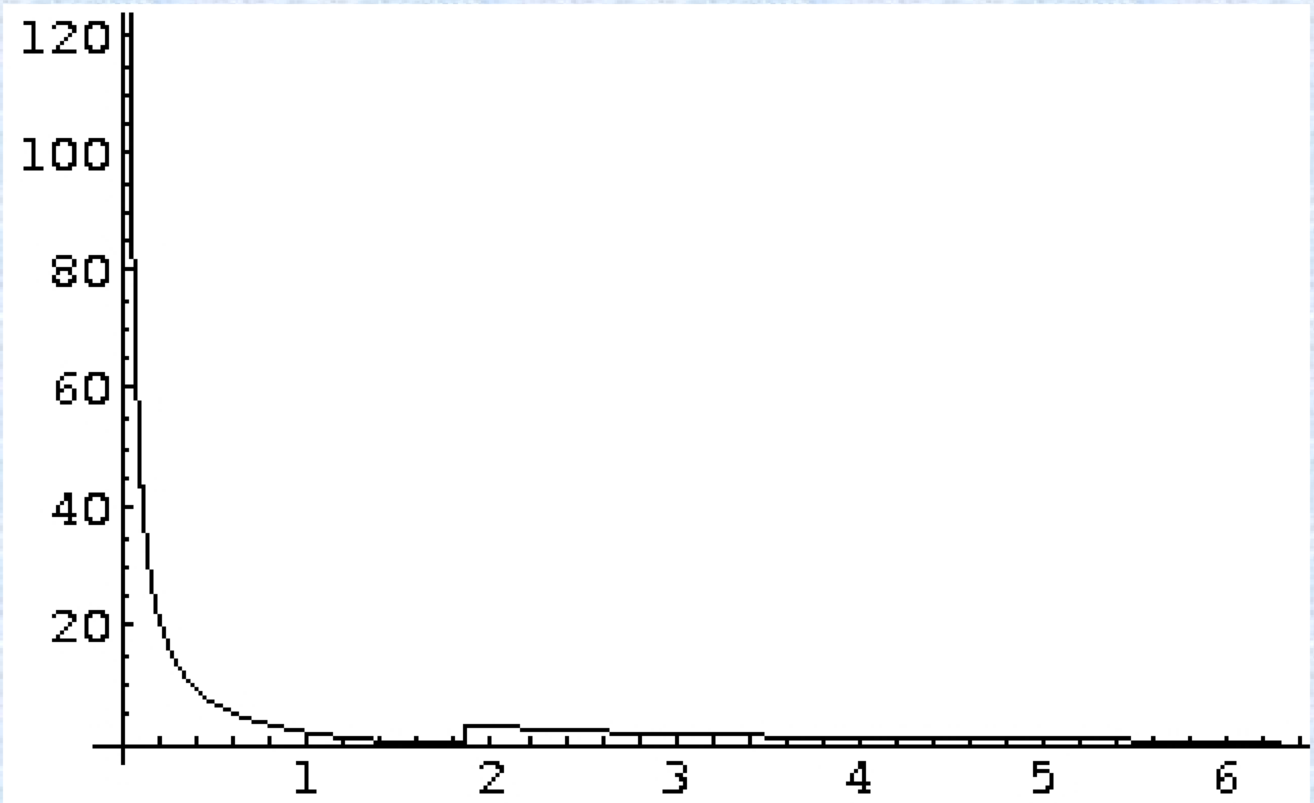
$$k_i - k_d\omega^2 = \sqrt{M(\omega)} \quad \text{for} \quad \omega \in \Omega^+$$



$\sqrt{M(\omega)}$ vs. ω with $k_p = 1$

- For $k_i - k_d\omega^2 = -\sqrt{M(\omega)} < 0,$

$\Omega^- = [1.35894, 1.8659] \cup [4.37326, +\infty)$



$L(\omega)$ vs. ω with $k_i - k_d\omega^2 = -\sqrt{M(\omega)}$

Finally, we exclude $S_{L,kp}{}^{+}$ and $S_{L,kp}{}^{-}$ from $S_{1,kp}$ to get $S_{R,kp}$

Stabilizing region of (k_i, k_d) with k_p=1 for plant with delay up to 1.