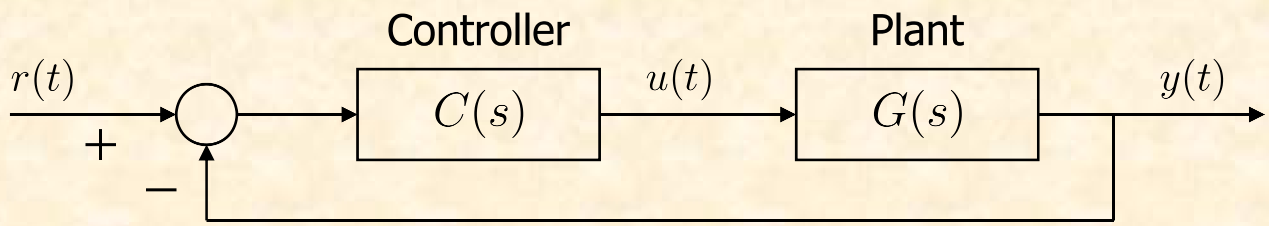
PID Controllers for Delay-Free LTI Systems

PID Design

- Typically, PID designs rely on adhoc tuning rules developed based on empirical observations and industrial experience.
- The state feedback observer based theory of modern control theory (H_2, H_∞, μ, l_1) optimal control cannot be applied to PID design.

Question:

For a given LTI system, determine whether stabilization by using PID controllers is possible.



$$G(s) = \frac{N(s)}{D(s)}, \qquad C(s) = \frac{sk_p + k_i + k_d s^2}{s(1+sT)}, \quad \text{for } T > 0$$

- The closed-loop characteristic polynomial is:

$$\delta(s, k_p, k_i, k_d) = s(1+sT)D(s) + \left(k_i + k_d s^2\right) N(s) + k_p s N(s)$$

Stabilization Problem using a PID Controller

To determine the values of k_p, k_i, k_d for which the closed-loop characteristic polynomial is Hurwitz.

STABILIZING SET

$$\mathbf{k} := [\ k_p,\ k_i,\ k_d\ ] \quad \text{and let} \quad \mathcal{S}^o := \{\mathbf{k} : \delta(s,k) \in \mathcal{H}\}$$

where $\mathcal{H}$ is the set of Hurwitz polynomials of the prescribed degree.

Advantages of obtaining stabilizing set

- Due to the presence of integral action on the error, any controller in the set provides asymptotic tracking and disturbance rejection for step inputs.
- A set of controller parameters satisfying additional requirements can be found as a subset of the stabilizing set.
- If the set is empty, it is not possible to stabilize the plant by a PID controller.

How to find $\mathcal{S}^o$

A naive application of the Routh-Hurwitz criterion to δ(s,**k**) leads to a description of $\mathcal{S}^o$ in terms of highly nonlinear and intractable inequalities.

SIGNATURE FORMULAS

Let p(s) denote a real polynomial of degree n without zeros on the imaginary axis.

$$p(s) := \underbrace{p_0 + p_2 s^2 + \cdots}_{p_{\text{even}}(s^2)} + s\,\underbrace{\left(p_1 + p_3 s^2 + \cdots\right)}_{p_{\text{odd}}(s^2)}$$

so that $$p(j\omega) = p_r(\omega) + j p_i(\omega)$$

where $$p_r(\omega) = p_{\text{even}}(-\omega^2), \quad p_i(\omega) = \omega p_{\text{odd}}(-\omega^2).$$

Definition

$$\mathrm{sgn}[x] = \begin{cases} -1 & \text{if} \quad x < 0 \\ 0 & \text{if} \quad x = 0 \\ 1 & \text{if} \quad x > 0. \end{cases}$$

Lemma

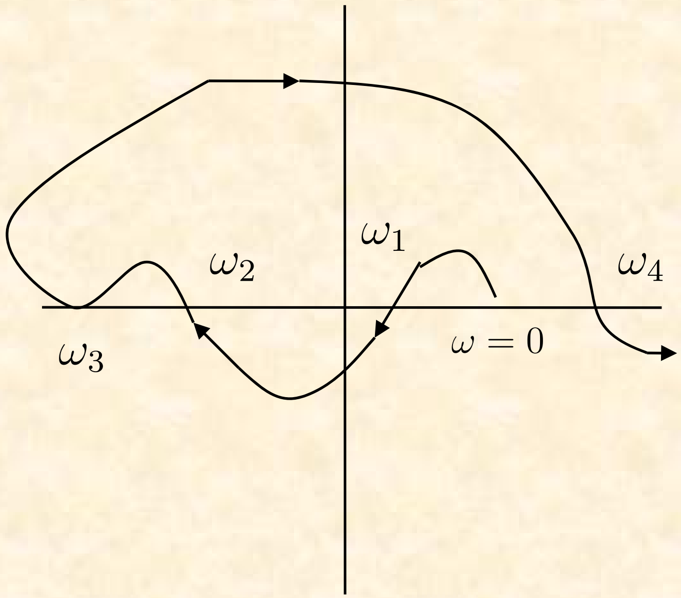
$$\Delta_0^\infty \angle p(j\omega) = \frac{\pi}{2}(l - r) =: \frac{\pi}{2}\sigma(p)$$

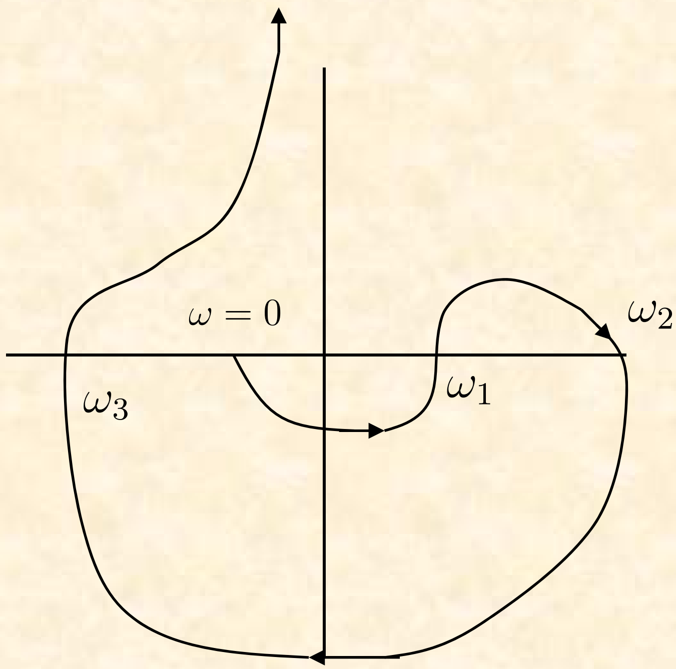
l: number of roots of $p(s)$ in $\mathcal{C}^-$

r: number of roots of $p(s)$ in $\mathcal{C}^+$

$\sigma(p)$: signature of p(s)

Computation of σ(p)





p(jω) for p(s) of even degree

p(jω) for p(s) of odd degree

$$
\begin{aligned}
\Delta_0^{\omega_1}\angle p(j\omega) &= \operatorname{sgn}[p_i(0^+)]\big(\operatorname{sgn}[p_r(0) - \operatorname{sgn}[p_r(\omega_1)]\big)\frac{\pi}{2} \\
\Delta_{\omega_1}^{\omega_2}\angle p(j\omega) &= \operatorname{sgn}[p_i(\omega_1^+)]\big(\operatorname{sgn}[p_r(\omega_1) - \operatorname{sgn}[p_r(\omega_2)]\big)\frac{\pi}{2} \\
\Delta_{\omega_2}^{\omega_3}\angle p(j\omega) &= \operatorname{sgn}[p_i(\omega_2^+)]\big(\operatorname{sgn}[p_r(\omega_2) - \operatorname{sgn}[p_r(\omega_3)]\big)\frac{\pi}{2} \\
\Delta_{\omega_3}^{\omega_4}\angle p(j\omega) &= \operatorname{sgn}[p_i(\omega_3^+)]\big(\operatorname{sgn}[p_r(\omega_3) - \operatorname{sgn}[p_r(\omega_4)]\big)\frac{\pi}{2} \\
\Delta_{\omega_4}^{+\infty}\angle p(j\omega) &= \operatorname{sgn}[p_i(\omega_4^+)]\big(\operatorname{sgn}[p_r(\omega_4) - \operatorname{sgn}[p_r(\infty)]\big)\frac{\pi}{2}
\end{aligned}
$$

and

$$
\begin{aligned}
\operatorname{sgn}[p_i(\omega_1^+)] &= -\operatorname{sgn}[p_i(0^+)] \\
\operatorname{sgn}[p_i(\omega_2^+)] &= -\operatorname{sgn}[p_i(\omega_1^+)] = +\operatorname{sgn}[p_i(0^+)] \\
\operatorname{sgn}[p_i(\omega_3^+)] &= +\operatorname{sgn}[p_i(\omega_2^+)] = +\operatorname{sgn}[p_i(0^+)] \\
\operatorname{sgn}[p_i(\omega_4^+)] &= -\operatorname{sgn}[p_i(\omega_3^+)] = -\operatorname{sgn}[p_i(0^+)]
\end{aligned}
$$

Note that 0, ω_1, ω_2, ω_4 are the real zeros of $p_i(\omega)$ of odd multiplicity whereas ω_3 is a real zero of even multiplicity.

$$\begin{aligned}\Delta_0^\infty \angle p(j\omega) &= \Delta_0^{\omega_1} \angle p(j\omega) + \Delta_{\omega_1}^{\omega_2} \angle p(j\omega) + \Delta_{\omega_2}^{\omega_4} \angle p(j\omega) + \Delta_{\omega_4}^{\infty} \angle p(j\omega) \\ &= \frac{\pi}{2}\mathrm{sgn}[p_i(0^+)]\big(\mathrm{sgn}[p_r(0)] - \mathrm{sgn}[p_r(\omega_1)]\big) - \mathrm{sgn}[p_i(0^+)]\big(\mathrm{sgn}[p_r(\omega_1)] - \mathrm{sgn}[p_r(\omega_2)]\big) \\ &\quad +\mathrm{sgn}[p_i(0^+)]\big(\mathrm{sgn}[p_r(\omega_2)] - \mathrm{sgn}[p_r(\omega_4)]\big) - \mathrm{sgn}[p_i(0^+)]\big(\mathrm{sgn}[p_r(\omega_4)] - \mathrm{sgn}[p_r(\infty)]\big).\end{aligned}$$

We have

$$\begin{aligned}\Delta_0^\infty \angle p(j\omega) &= \frac{\pi}{2}\mathrm{sgn}[p_i(0^+)]\big(\mathrm{sgn}[p_r(0)] - 2\mathrm{sgn}[p_r(\omega_1)] + 2\mathrm{sgn}[p_r(\omega_2)] \\ &\qquad -2\mathrm{sgn}[p_r(\omega_4)] + \mathrm{sgn}[p_r(\infty)]\big).\end{aligned}$$

From this observation, we have the following theorem.

Theorem

Let p(s) be a polynomial with real coefficients, of degree n, without zeros on the imaginary axis. Write

$$p(j\omega) = p_r(\omega) + jp_i(\omega)$$

and let ω_0, ω_1, ω_3, …, ω_{l-1} denote the real nonnegative zeros of $p_i(\omega)$ with odd multiplicities with $\omega_0=0$. Then

If n is even

$$\sigma(p) = \mathrm{sgn}[p_i(0^+)]\left(\mathrm{sgn}[p_r(0)] + 2\sum_{j=1}^{l-1}(-1)^j\mathrm{sgn}[p_r(\omega_j)] + (-1)^l\mathrm{sgn}[p_r(\infty)]\right).$$

If n is odd

$$\sigma(p) = \mathrm{sgn}[p_i(0^+)]\left(\mathrm{sgn}[p_r(0)] + 2\sum_{j=1}^{l-1}(-1)^j\mathrm{sgn}[p_r(\omega_j)]\right).$$

COMPUTATION OF STABILIZING SET

- Plant and Controller

$$P(s) = \frac{N(s)}{D(s)}, \qquad C(s) = \frac{k_p s + k_i + k_d s^2}{s(1+sT)}, \quad T > 0.$$

where $\deg[D(s)] = n > \deg[N(s)] = m$

- The closed-loop characteristic polynomial

$$\delta(s) = sD(s)(1+sT) + \left(k_p s + k_i + k_d s^2\right) N(s)$$

- Define $\nu(s) := \delta(s)N(-s)$

$$\nu(s) = \nu_{\text{even}}(s^2, k_i, k_d) + s\nu_{\text{odd}}(s^2, k_p)$$

Theorem

The closed-loop system is stable if and only if

$$\sigma(\nu) = n - m + 2 + 2z^{+}.$$

Proof

Closed-loop stability is equivalent to the requirement that the n+2 zeros of δ(s) lie in the open LHP. This is equivalent to

$$\sigma(\delta) = n + 2$$

and to

$$\begin{aligned} \sigma(\nu) &= n + 2 + z^{+} - z^{-} \\ &= n + 2 + z^{+} - (m - z^{+}) \\ &= (n - m) + 2 + 2z^{+}. \end{aligned}$$

Procedure of Calculating Stabilizing Set

Step 1: Fix $k_p = k_p^*$ and let $0 < \omega_1 < \omega_2 < \ldots < \omega_{l-1}$ denote the real, positive, finite frequencies which are zeros of

$$\nu_{\text{odd}}(-\omega^2, k_p^*) = 0$$

of odd multiplicities. Let $\omega_0 := 0$ and $\omega_l := \infty$.

Step 2: Write $j = \text{sgn}\left[v_{\text{odd}}(0, k_p^*)\right]$

and determine strings of integers, i_0, i_1, … such that:

If n+m is even

$$j\left(i_0 - 2i_1 + 2i_2 + \cdots + (-1)^{l-1} 2i_{l-1} + (-1)^l i_l\right) = n - m + 2 + 2z^+$$

If n+m is odd

$$j\left(i_0 - 2i_1 + 2i_2 + \cdots + (-1)^{l-1} 2i_{l-1}\right) = n - m + 2 + 2z^+$$

Step 3: Let I_1, I_2, I_3, … denote distinct strings $\{i_0, i_1, \ldots\}$ satisfying the expression in step 2. Then the stabilizing sets in k_i, k_d space, for $k_p = k_p^*$ are given by the linear inequalities

$$\nu_r\left(-\omega_t^2, k_i, k_d\right) i_t > 0$$

where the i_t range over each of the string I_1, I_2, ….

Step 4: For each string I_j, the expression in step 3 generates a convex stability set $\mathcal{S}_j(k_p^*)$ and the complete set for fixed k_p^* is the union of these convex sets

$$\mathcal{S}(k_p^*) = \cup_j \mathcal{S}_j(k_p^*).$$

Step 5: The complete stabilizing set in (k_p, k_i, k_d) space can be found by sweeping k_p over the real axis and repeating the steps 1-4.

NOTE:

- It is easy to see that the range of sweeping k_p can be restricted to those values such that the number of roots *ℓ-1* can satisfy the expressions in Step 2.
- From this consideration, we found that kp needs to be swept over those ranges where $\nu_{\text{odd}}(-\omega^2, k_p^*) = 0$ is satisfied with *ℓ-1* given by:

$l - 1 \geq \dfrac{n - m + 2z^+}{2}$	for *n+m* even
$l - 1 \geq \dfrac{n - m + 1 + 2z^+}{2}$	for *n+m* odd

Remark: If the PID controller with pure derivative action is used (T=0), it is easy to see that the signature requirement for stability becomes

$$\sigma(\nu) = n - m + 1 + 2z^+.$$

Example: Determine stabilizing PID gains for the plant:

$$P(s) = \frac{N(s)}{D(s)} = \frac{s^3 - 2s^2 - s - 1}{s^6 + 2s^5 + 32s^4 + 26s^3 + 65s^2 - 8s + 1}.$$

- Using T=0, the closed-loop characteristic polynomial is

$$\delta(s, k_p, k_i, k_d) = sD(s) + (k_i + k_d s^2)N(s) + k_p s N(s).$$

- Here *n=6* and *m=3*, we have

$$N_e(s^2) = -2s^2 - 1, \qquad N_o(s^2) = s^2 - 1,$$
$$D_e(s^2) = s^6 + 32s^4 + 65s^2 + 1, \qquad D_o(s^2) = 2s^4 + 26s^2 - 8,$$

and

$$\begin{aligned} \nu(s) &= \delta(s, k_p, k_i, k_d)N(-s) \\ &= \left[s^2\left(-s^8 - 35s^6 - 87s^4 + 54s^2 + 9\right) + \left(k_i + k_d s^2\right)\left(-s^6 + 6s^4 + 3s^2 + 1\right)\right] \\ &\quad + s\left[\left(-4s^8 - 89s^6 - 128s^4 - 75s^2 - 1\right) + k_p\left(-s^6 + 6s^4 + 3s^2 + 1\right)\right] \end{aligned}$$

- $$\nu(j\omega, k_p, k_i, k_d) = \left[p_1(\omega) + \left(k_i - k_d\omega^2\right) p_2(\omega)\right] + j\left[q_1(\omega) + k_p q_2(\omega)\right]$$

where

$$\begin{aligned} p_1(\omega) &= \omega^{10} - 35\omega^8 + 87\omega^6 + 54\omega^4 - 9\omega^2 \\ p_2(\omega) &= \omega^6 + 6\omega^4 - 3\omega^2 + 1 \\ q_1(\omega) &= -4\omega^9 + 89\omega^7 - 128\omega^5 + 75\omega^3 - \omega \\ q_2(\omega) &= \omega^7 + 6\omega^5 - 3\omega^3 + \omega. \end{aligned}$$

- We find that $z^+=1$ so it is required for stability that

$$\sigma(\nu) = n - m + 1 + 2z^+ = 6.$$

- Since deg[ν(s)] is even, $q(\omega)$ must have at least two positive real roots of odd multiplicity. So the allowable range for k_p is (-24.7513, 1). Pick k_p=-18.

$$\begin{aligned} q(\omega, -18) &= q_1(\omega) - 18q_2(\omega) \\ &= -4\omega^9 + 71\omega^7 - 236\omega^5 + 129\omega^3 - 19\omega. \end{aligned}$$

- The real, non-negative, distinct finite zeros of $q(\omega,-18)$ with odd multiplicities are

$$\omega_0 = 0, \quad \omega_1 = 0.5195, \quad \omega_2 = 0.6055, \quad \omega_3 = 1.8804, \quad \omega_4 = 3.6848.$$

- Define $\omega_5=\infty$. $\quad \mathrm{sgn}[q(0,-18)] = -1$

- It follows that every admissible string must satisfy

$$\{i_0 - 2i_1 + 2i_2 - 2i_3 + 2i_4 - i_5\} \cdot (-1) = 6$$

and the admissible strings are

$$\begin{aligned}
\mathcal{I}_1 &= \{-1,-1,-1,1,-1,1\} \\
\mathcal{I}_2 &= \{-1,1,1,1,-1,1\} \\
\mathcal{I}_3 &= \{-1,1,-1,-1,-1,1\} \\
\mathcal{I}_4 &= \{-1,1,-1,1,1,1\} \\
\mathcal{I}_5 &= \{1,1,-1,1,-1,-1\}.
\end{aligned}$$

- For I_t, it follows that the stabilizing (k_i, k_d) values corresponding to k_p=-18 must satisfy the string of inequalities:

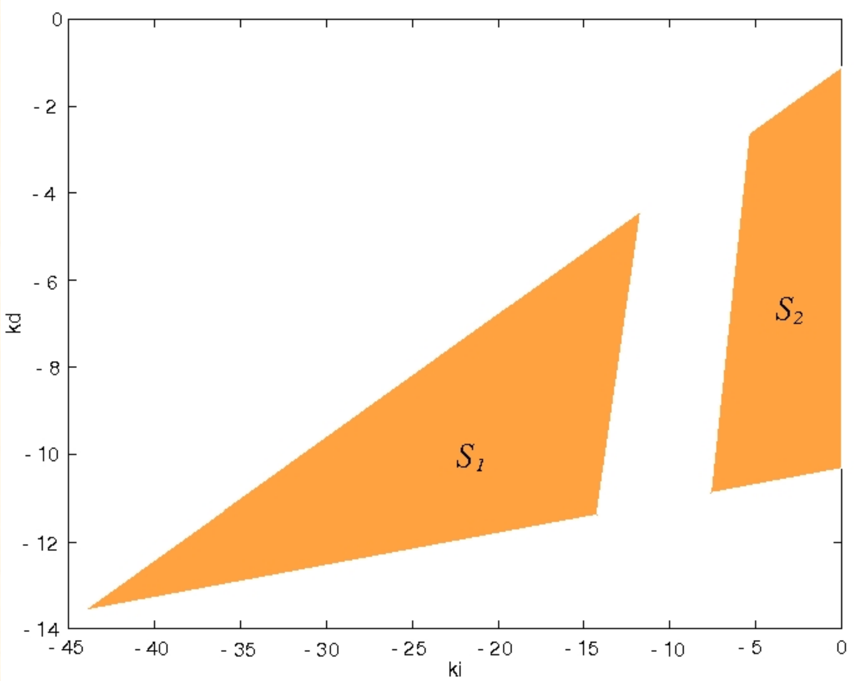
$$\begin{aligned}
p_1(\omega_0) + \left(k_i - k_d\omega_0^2\right) p_2(\omega_0) &< 0\\
p_1(\omega_1) + \left(k_i - k_d\omega_1^2\right) p_2(\omega_1) &< 0\\
p_1(\omega_2) + \left(k_i - k_d\omega_2^2\right) p_2(\omega_2) &< 0\\
p_1(\omega_3) + \left(k_i - k_d\omega_3^2\right) p_2(\omega_3) &> 0\\
p_1(\omega_4) + \left(k_i - k_d\omega_4^2\right) p_2(\omega_4) &< 0\\
p_1(\omega_5) + \left(k_i - k_d\omega_5^2\right) p_2(\omega_5) &> 0
\end{aligned}$$

- Substituting for ω_0, ω_1, ω_2, ω_3, ω_4, ω_5, we have

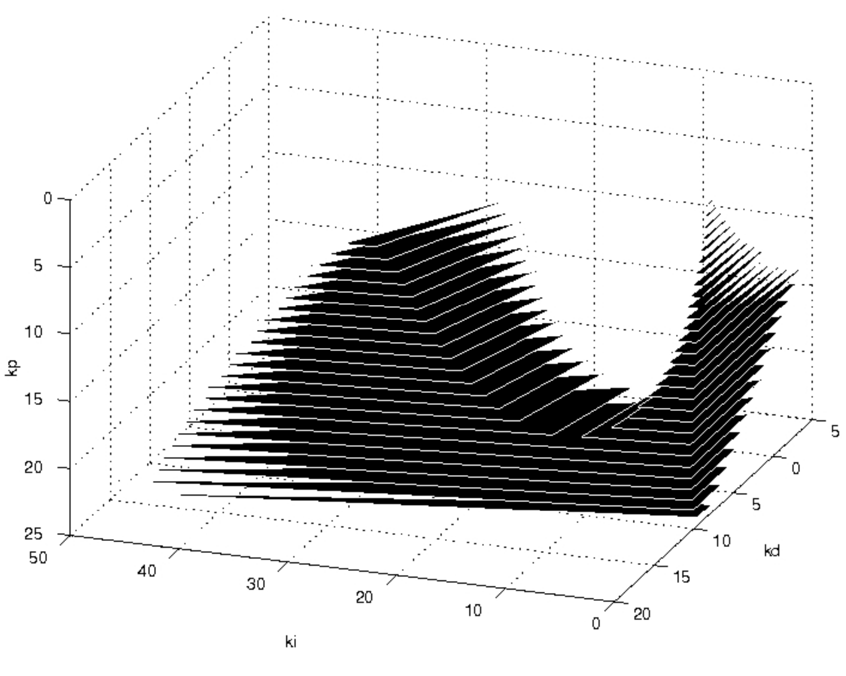
$$\mathcal{S}_1 : \left\{ \begin{aligned}
k_i &< 0\\
k_i - 0.2699k_d &< -4.6836\\
k_i - 0.3666k_d &< -10.0797\\
k_i - 3.5358k_d &> 3.912\\
k_i - 13.5777k_d &< 140.2055
\end{aligned} \right.$$

$$\mathcal{S}_2 : \left\{ \begin{aligned}
k_i &< 0\\
k_i - 0.2699k_d &> -4.6836\\
k_i - 0.3666k_d &> -10.0797\\
k_i - 3.5358k_d &> 3.912\\
k_i - 13.5777k_d &< 140.2055
\end{aligned} \right.$$

$$\mathcal{S}_3 = \mathcal{S}_4 = \mathcal{S}_5 = \emptyset$$



The stabilizing set of (k_i, k_d) when k_p=-18



The stabilizing set of (k_p, k_i, k_d)

PID DESIGN WITH PERFORMANCE REQUIREMENTS

Signature Formulas for Complex Polynomials

Consider the polynomial c(s) with complex coefficients and let c(s) have no *jω* axis zeros. Let

$$c(j\omega) = p(\omega) + jq(\omega)$$

where p(ω) and q(ω) are polynomials with real coefficients. Write

$$j_- = \mathrm{sgn}[q(-\infty)], \quad j_+ = \mathrm{sgn}[q(\infty)] \quad \text{and} \quad i_k = \mathrm{sgn}[p(\omega_k)]$$

Theorem

If deg[p]>deg[q]

$$\begin{aligned}\sigma(c) &= \frac{1}{2} j_- \left\{ i_0 - 2i_1 + 2i_2 + \cdots + (-1)^{l-1} 2i_{l-1} + (-1)^l i_l \right\} \\ &= \frac{1}{2} j_+ (-1)^{l-1} \left\{ i_0 - 2i_1 + \cdots + (-1)^{l-1} 2i_{l-1} + (-1)^l i_l \right\}\end{aligned}$$

If deg[q]≥deg[p]

$$\begin{aligned}\sigma(c) &= j_- \left\{ -i_1 + i_2 - i_3 + \cdots + (-1)^{l-1} i_{l-1} \right\} \\ &= j_+ (-1)^{l-1} \left\{ -i_1 + i_2 - i_3 + \cdots + (-1)^{l-1} i_{l-1} \right\}.\end{aligned}$$

Proof: For deg[p]>deg[q], c($j\omega$) approaches the real axis as $|\omega|$ goes to ∞. Thus,

$$\Delta_{-\infty}^{+\infty}\angle c(j\omega) = \Delta_{-\infty}^{\omega_1}\angle c(j\omega) + \Delta_{\omega_1}^{\omega_2}\angle c(j\omega) + \cdots + \Delta_{\omega_{l-1}}^{+\infty}\angle c(j\omega)$$

and

$$\begin{aligned} \Delta_{-\infty}^{\omega_1}\angle c(j\omega) &= \frac{\pi}{2} j_- \left(i_0 - i_1\right) \\ \Delta_{\omega_k}^{\omega_{k+1}}\angle c(j\omega) &= \frac{\pi}{2} j_- (-1)^k \left(i_k - i_{k+1}\right), \quad k = 0, 1, \cdots, l-1 \end{aligned}$$

Using $\Delta_{-\infty}^{+\infty}\angle c(j\omega) = \pi(l - r), \quad \sigma(c) := l - r$

$$\sigma(c) = \frac{1}{2} j_- \left\{ i_0 - 2i_1 + 2i_2 + \cdots + (-1)^{l-1} 2 i_{l-1} + (-1)^l i_l \right\}$$

From $j_- = j_+ (-1)^{l-1}$,

$$\sigma(c) = \frac{1}{2} j_+ (-1)^{l-1} \left\{ i_0 - 2i_1 + \cdots + (-1)^{l-1} 2 i_{l-1} + (-1)^l i_l \right\}$$

The case deg[q]≥deg[p] may be proved similarly.

Complex PID Stabilization Algorithm

- Consider $$c(s, k_p, k_i, k_d) = L(s) + \left(k_d s^2 + k_p s + k_i\right) M(s)$$

- Define $$\nu(j\omega) = c(j\omega, k_p, k_i, k_d)M^*(j\omega) = p(\omega, k_i, k_d) + jq(\omega, k_p)$$

  where
$$\begin{aligned}
p(\omega, k_i, k_d) &= p_1(\omega) + (k_i - k_d\omega^2)p_2(\omega) \\
q(\omega, k_p) &= q_1(\omega) + k_p q_2(\omega) \\
p_1(\omega) &= L_r(j\omega)M_r(j\omega) - L_i(j\omega)M_i(j\omega) \\
p_2(\omega) &= M_r^2(j\omega) - M_i^2(j\omega) \\
q_1(\omega) &= \frac{1}{j}\left[L_i(j\omega)M_r(j\omega) - L_r(j\omega)M_i(j\omega)\right] \\
q_2(\omega) &= \omega\left[M_r^2(j\omega) - M_i^2(j\omega)\right]
\end{aligned}$$

Step 1: Compute $p_1(\omega)$, $p_2(\omega)$, $q_1(\omega)$, $q_2(\omega)$

Step 2: Determine the allowable ranges of k_p such that $q(\omega,k_p)$ has at least

$$\begin{cases} |n-(l(M(s))-r(M(s)))| & \text{if } \deg[q] \geq \deg[p] \\ |n-(l(M(s))-r(M(s)))|-1 & \text{if } \deg[p]) > \deg[q] \end{cases}$$

real, distinct finite zeros with odd multiplicities.

Step 3: For fixed $k_p=k_p^*$, solve the real, distinct finite zeros of $q(\omega,k_p^*)$ with odd multiplicities and denote them by $\omega_1<\omega_2<\ldots<\omega_{l-1}$ and let $\omega_0=-\infty$ and $\omega_l=\infty$;

Step 4: Find sequences of integers i_0, i_1, i_2, …, il with

$$i_t \in \{-1, 1\}, \text{ for all other } t = 0, 1, \cdots, l.$$

such that

$$n - (l(M(s)) - r(M(s))) =$$

$$\begin{cases} \frac{1}{2}\left\{i_0 \cdot (-1)^{l-1} + 2\sum_{r=1}^{l-1} i_r \cdot (-1)^{l-1-r} - i_l\right\} \cdot \operatorname{sgn}[q(\infty, k_p)] & \text{if } \deg[p] > \deg[q] \\ \frac{1}{2}\left\{2\sum_{r=1}^{l-1} i_r \cdot (-1)^{l-1-r}\right\} \cdot \operatorname{sgn}[q(\infty, k_p)] & \text{if } \deg[q] \geq \deg[p] \end{cases}$$

Step 5: The stabilizing sets in k_i, k_d space are given by

$$p(\omega_t, k_i.k_d) i_t > 0$$

where i_t are taken from the admissible strings satisfying the signature condition for stability.

Step 6: Repeat the procedure by updating k_p in the admissible range.

PID Design with Guaranteed Gain and Phase Margins

Let A_m and θ_m denote the desired gain and phase margins. Then the PID gain value (k_p, k_i, k_d) must satisfy the following conditions.

$$1. \quad sD(s) + A\left(k_d s^2 + k_p s + k_i\right) N(s) \in \mathcal{H} \ \text{ for all } A \in [1, A_m]$$

$$2. \quad sD(s) + e^{-j\theta}\left(k_d s^2 + k_p s + k_i\right) N(s) \in \mathcal{H} \ \text{ for all } \theta \in [0, \theta_m]$$

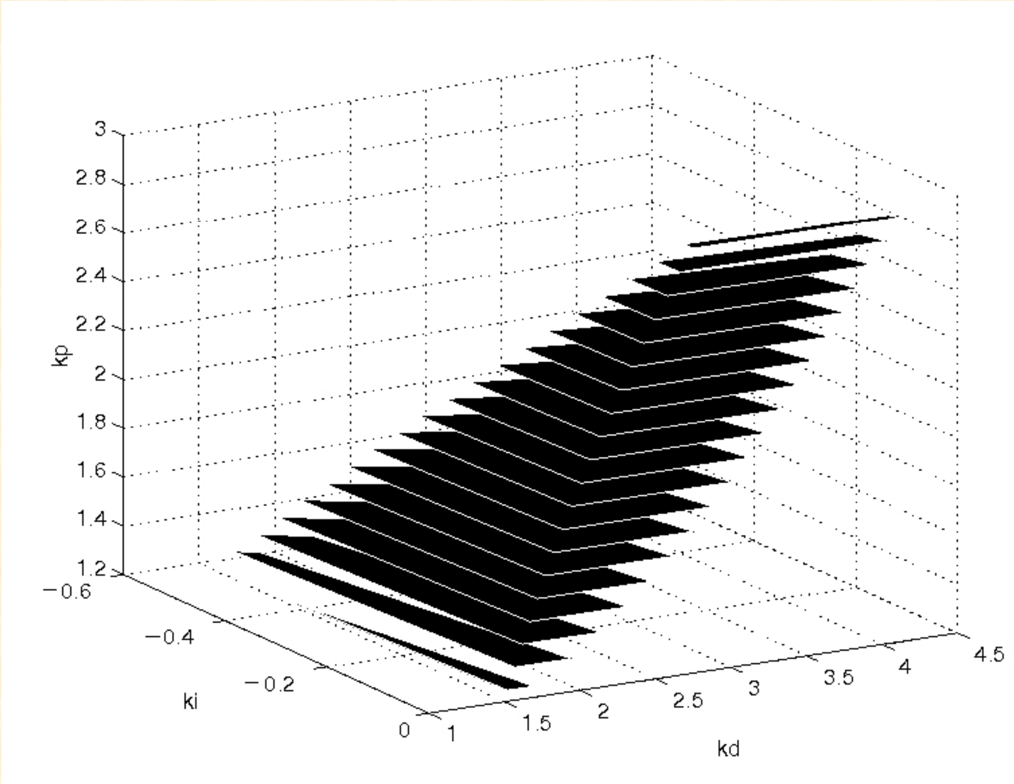
Example

$$G(s) = \frac{N(s)}{D(s)} = \frac{2s-1}{s^4 + 3s^3 + 4s^2 + 7s + 9}$$

Determining PID gains that provide a gain margin Am≥3.0 and a phase margin $\theta_m \geq 40^o$ is equivalent to find PID values satisfying

$$1. \quad s\left(s^4 + 3s^3 + 4s^2 + 7s + 9\right) + A\left(k_d s^2 + k_p s + k_i\right)(2s-1) \in \mathcal{H} \ \text{ for all } A \in [1, 3.0]$$

$$2. \quad s\left(s^4 + 3s^3 + 4s^2 + 7s + 9\right) + e^{-j\theta}\left(k_d s^2 + k_p s + k_i\right)(2s-1) \in \mathcal{H} \ \text{ for all } \theta \in [0^o, 40^o]$$



The set (k_p, k_i, k_d) values for which the resulting closed-loop system achieves a gain margin greater than 3.0 and a phase margin greater than 40 degree.

PID Design with an H_∞ criterion

Find PID controller for which the closed-loop system is internally stable and the H_∞ norm of a certain closed-loop transfer function is less than a prescribed level.

- The sensitivity function $$S(s) = \frac{1}{1 + C(s)G(s)}$$

- The complementary sensitivity function $$T(s) = \frac{C(s)G(s)}{1 + C(s)G(s)}$$

- The input sensitivity function $$U(s) = \frac{C(s)}{1 + C(s)G(s)}$$

- All these transfer functions can be represented in the following general form:

$$T_{cl}(s, k_p, k_i, k_d) = \frac{A(s) + (k_d s^2 + k_p s + k_i)\, B(s)}{sD(s) + (k_d s^2 + k_p s + k_i)\, N(s)}$$

where A(s) and B(s) are some real polynomials.

- For the transfer function Tcl(s,k_p,k_i,k_d) and a given $\gamma > 0$, the standard H_∞ performance specification usually takes the form:

$$\|W(s) T_{cl}(s, k_p, k_i, k_d)\|_\infty < \gamma, \qquad W(s) = \frac{W_n(s)}{W_d(s)}$$

where W(s) is a stable frequency-dependent weighting function that is selected to capture the desired design objectives.

- Define

$$\delta(s,k_p,k_i,k_d) \triangleq sD(s) + \left(k_i + k_p s + k_d s^2\right) N(s)$$

$$\phi(s,k_p,k_i,k_d,\gamma,\theta) \triangleq \left[sW_d(s)D(s) + \frac{1}{\gamma}e^{j\theta}W_n(s)A(s)\right] + \left(k_d s^2 + k_p s + k_i\right)\left[W_d(s)N(s) + \frac{1}{\gamma}e^{j\theta}W_n(s)B(s)\right].$$

- For a given $\gamma > 0$ there exist PID gains such that

$$\|W(s)T_{cl}(s,k_p,k_i,k_d)\|_\infty < \gamma$$

if and only if the following conditions hold:

$$(1) \qquad \delta(s,k_p,k_i,k_d) \in \mathcal{H};$$

$$(2) \qquad \phi(s,k_p,k_i,k_d,\gamma,\theta) \in \mathcal{H} \text{ for all } \theta \in [0,2\pi);$$

$$(3) \qquad \|W(\infty)T_{cl}(\infty,k_p,k_i,k_d)\| < \gamma$$

Example

- Plant and controller

$$G(s) = \frac{N(s)}{D(s)} = \frac{s-1}{s^2+0.8s-0.2}, \qquad C(s) = \frac{k_d s^2 + k_p s + k_i}{s}$$

- Determine all stabilizing PID gain values for which

$$\|W(s)T(s,k_p,k_i,k_d)\|_\infty < 1,$$

where $T(s,k_p.k_i,k_d)$ is the complementary sensitivity function:

$$T(s,k_p,k_i,k_d) = \frac{(k_d s^2 + k_p s + k_i)(s-1)}{s(s^2+0.8s-0.2) + (k_d s^2 + k_p s + k_i)(s-1)}$$

and the weight W(s) is chosen as a high pass filter:

$$W(s) = \frac{s+0.1}{s+1}$$

Solution

(k_p,k_i,k_d) values meeting the H_∞ performance constraints exist iff the following conditions hold:

$$
\begin{aligned}
&(1) \qquad \delta(s,k_p,k_i,k_d) = s(s^2+0.8s-0.2)+(k_d s^2+k_p s+k_i)(s-1) \ \in \mathcal{H};\\
&(2) \qquad \phi(s,k_p,k_i,k_d,1,\theta) = s(s+1)(s^2+0.8s-0.2)\\
&\qquad\qquad + \left(k_d s^2+k_p s+k_i\right)\left[(s+1)(s-1)+e^{j\theta}(s+0.1)(s-1)\right] \ \in \mathcal{H} \text{ for all } \theta\in[0,2\pi];\\
&(3) \qquad |W(\infty)T(\infty,k_p,k_i,k_d)| = \left|\frac{k_d}{k_d+1}\right| < 1.
\end{aligned}
$$

- For the condition (1), with a fixed k_p=-0.35, determine all values of (k_i,k_d) by using the standard algorithm with

$$L(s) = s\left(s^2+0.8s-0.2\right) \quad \text{and} \quad M(s) = s-1$$

This set is denoted by $\mathcal{S}_{(1,-0.35)}$

- For the condition (2), fixing k_p and any fixed $\theta \in [0, 2\pi),$ by setting

$$\begin{aligned} L(s) &= s(s+1)\left(s^2+0.8s-0.2\right), \\ M(s,\theta) &= (s+1)(s-1)+e^{j\theta}(s+0.1)(s-1) \end{aligned}$$

  and using the complex stabilization algorithm we can determine the set of (k_i,k_d) values. Let this set be denoted by $\mathcal{S}_{(2,-0.35,\theta)}$

- By keeping k_p fixed, sweeping over $\theta \in [0, 2\pi)$ at each stage, we can determine the set for which condition (2) is satisfied. This set is denoted by (K_p=-0.35)

$$\mathcal{S}_{(2,-0.35)} = \cap_{\theta\in[0,2\pi)}\mathcal{S}_{(2,-0.35,\theta)}$$

- For the condition (3), we have

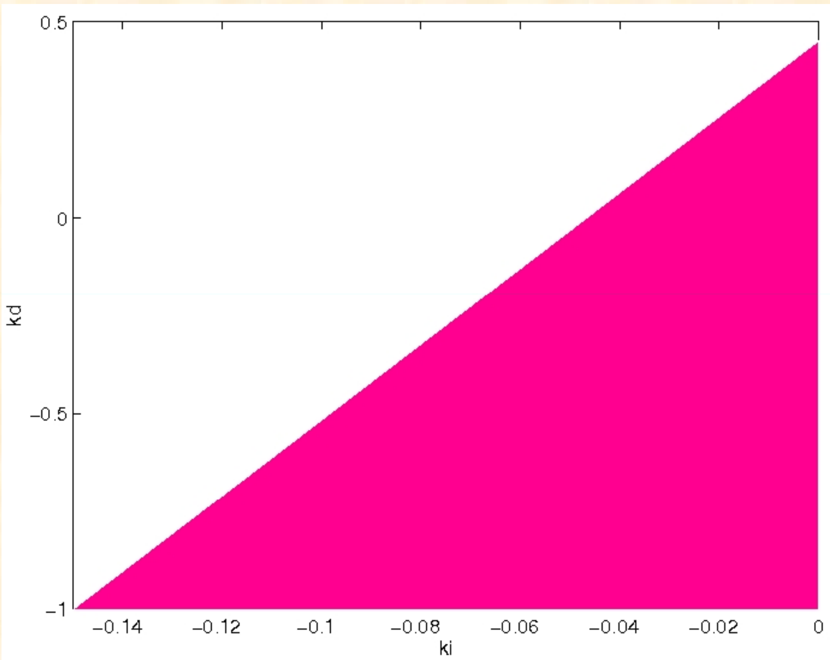
$$\mathcal{S}_{(3,-0.35)} = \{(k_i, k_d) : k_i \in \mathcal{R}, k_d > -0.5\}$$

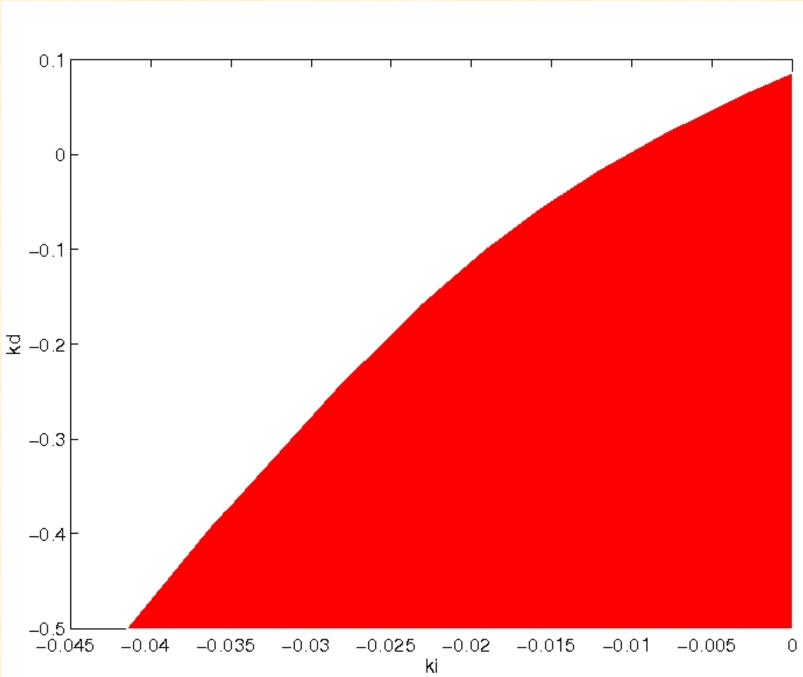
- For k_p=-0.35, the set of (k_i,k_d) values is

$$\mathcal{S}_{(-0.35)} = \cap_{i=1,2,3} \mathcal{S}_{(i,-0.35)}$$

- Using root loci, it is determined that a necessary condition for the existence of stabilizing (k_i,k_d) values is $k_p \in (-0.5566, -0.2197)$.

- By sweeping over this range of k_p and repeating the procedure, we have the set (k_p,k_i,k_d).

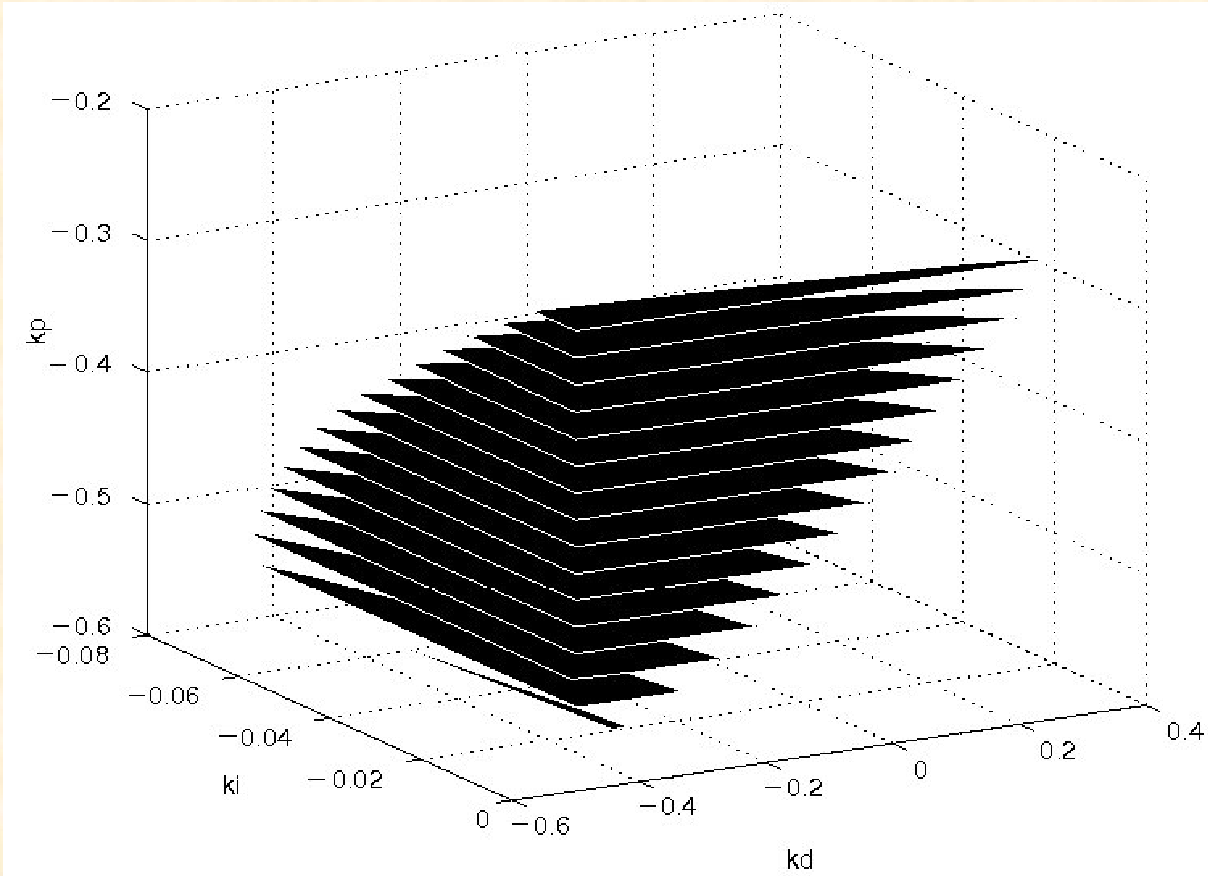




The set $\mathcal{S}_{(1,-0.35)}$

The set

$$\mathcal{S}_{(2,-0.35)} = \cap_{\theta\in[0,2\pi)} \mathcal{S}_{(2,-0.35,\theta)}$$



The set of stabilizing (k_p,k_i,k_d) values for which

$$|W(s)T(s, k_p, k_i, k_d)|_\infty < 1$$

PID Design for H_∞ Robust Performance

- Consider the following robust performance specification:

$$\| |W_1(s)S(s)| + |W_2(s)T(s)| \|_\infty < 1$$

where

$$W_1(s) = \frac{N_{W_1}(s)}{D_{W_1}(s)} \quad \text{and} \quad W_2(s) = \frac{N_{W_2}(s)}{D_{W_2}(s)}$$

are stable weighting functions, and S(s) and T(s) are the sensitivity and the complementary sensitivity functions, respectively.

- The characteristic polynomial is

$$\delta(s, k_p, k_i, k_d) \triangleq sD(s) + \left(k_i + k_p s + k_d s^2\right) N(s)$$

- Define the complex polynomial

$$\psi(s,k_p,k_i,k_d,\theta,\phi) \stackrel{\Delta}{=} sD_{W_1}(s)D_{W_2}(s)D(s)+e^{j\theta}sN_{W_1}(s)D_{W_2}(s)D(s) + \left(k_d s^2+k_p s+k_i\right)\cdot\left[D_{W_1}(s)D_{W_2}(s)N(s)+e^{j\phi}D_{W_1}(s)N_{W_2}(s)N(s)\right]$$

- The problem of synthesizing PID controllers for robust performance can be converted in to problem of determining (kp,ki,kd) for which the following conditions hold:

$$\begin{aligned}
&(1) \quad \delta(s,k_p,k_i,k_d)\in\mathcal{H};\\
&(2) \quad \psi(s,k_p,k_i,k_d,\theta,\phi) \in\mathcal{H} \text{ for all } \theta\in[0,2\pi) \text{ and for all } \phi\in[0,2\pi);\\
&(3) \quad |W_1(\infty)S(\infty)|+|W_2(\infty)T(\infty)|<1.
\end{aligned}$$

- The performance specification

$$\left\||W_1(s)S(s)|+|W_2(s)T(s)|\right\|_\infty<1$$

Example

- Plant $$G(s) = \frac{N(s)}{D(s)} = \frac{s-15}{s^2+s-1}$$

- The sensitivity and complementary sensitivity functions are

$$\begin{aligned} S(s, k_p, k_i, k_d) &= \frac{s\left(s^2+s-1\right)}{s\left(s^2+s-1\right)+\left(k_d s^2+k_p s+k_i\right)(s-15)}, \\ T(s, k_p, k_i, k_d) &= \frac{\left(k_d s^2+k_p s+k_i\right)(s-15)}{s\left(s^2+s-1\right)+\left(k_d s^2+k_p s+k_i\right)(s-15)}. \end{aligned}$$

- The weighting functions are chosen as

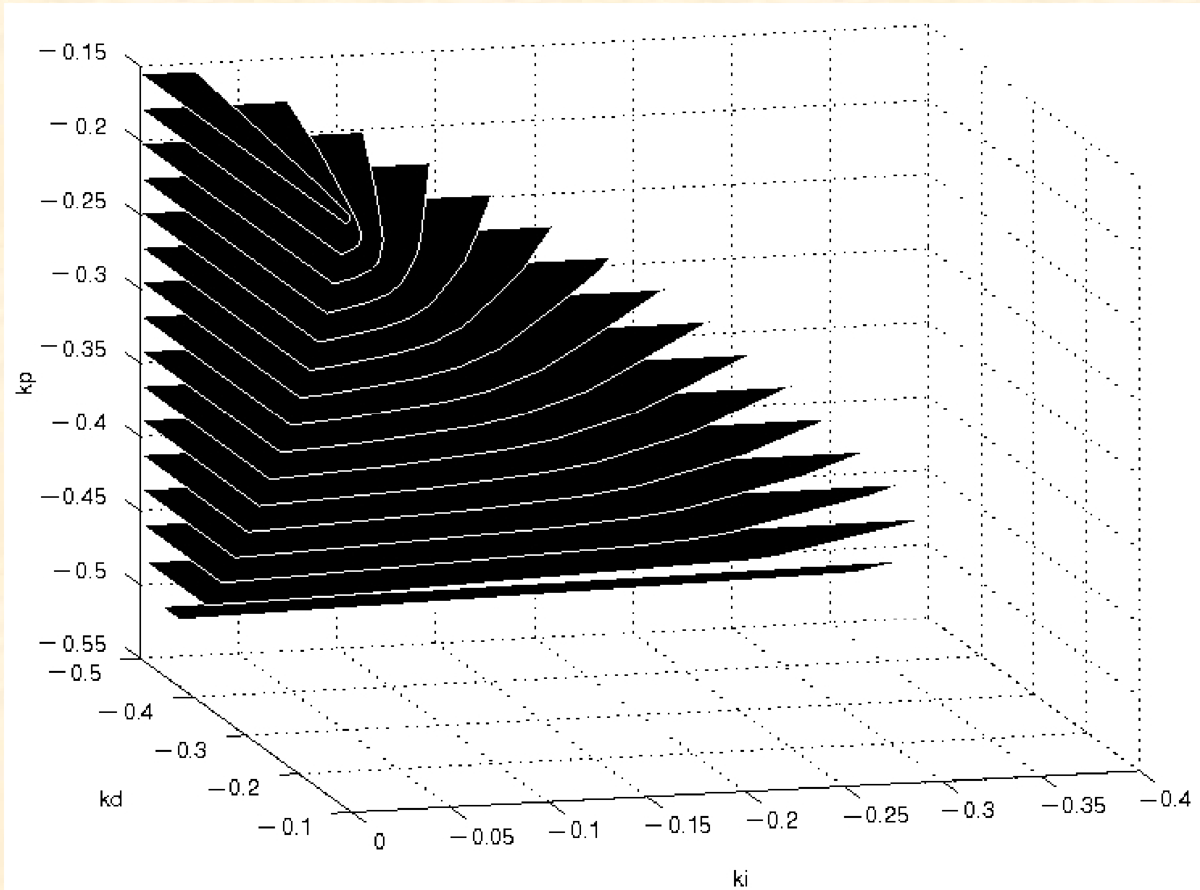
$$W_1(s) = \frac{0.2}{s+0.2} \quad \text{and} \quad W_2(s) = \frac{s+0.1}{s+1}$$

- The stabilizing (k_p, k_i, k_d) values meeting the performance specification exist iff the following conditions hold:

$$(1) \qquad \delta(s,k_p,k_i,k_d) = s\left(s^2+s-1\right) + \left(k_d s^2 + k_p s + k_i\right)(s-15) \in \mathcal{H};$$

$$\begin{aligned}(2) \qquad \psi(s,k_p,k_i,k_d,\theta,\phi) &= s(s+0.2)(s+1)\left(s^2+s-1\right) \\ &+e^{j\theta}s(0.2)(s+1)\left(s^2+s-1\right) + \left(k_d s^2 + k_p s + k_i\right) \\ &\cdot\left[(s+0.2)(s+1)(s-15) + e^{j\phi}(s+0.2)(s+0.1)(s-15)\right] \quad \in \mathcal{H} \\ &\text{for all } \theta \in [0,2\pi) \text{ and for all } \phi \in [0,2\pi);\end{aligned}$$

$$(3) \qquad |W_1(\infty)S(\infty,k_p,k_i,k_d)| + |W_2(\infty)T(\infty,k_p,k_i,k_d)| = \left|\frac{k_d}{k_d+1}\right| < 1.$$



The set of (k_p, k_i, k_d) values for which

$$\| |W_1(s)S(s, k_p, k_i, k_d)| + |W_2(s)T(s, k_p, k_i, k_d)| \|_\infty < 1$$