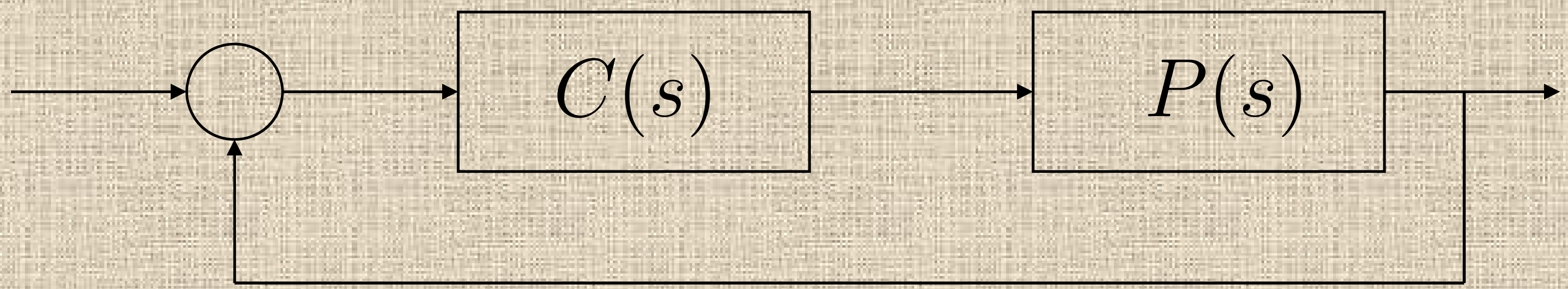
First Order Controllers for LTI Systems



- Plant and Controller: $P(s) := \dfrac{N(s)}{D(s)}$ and $C(s) := \dfrac{x_1 s + x_2}{s + x_3}$
- Using the standard even-odd decomposition of polynomials

$$\begin{aligned} N(s) &:= N_{\mathrm{e}}\left(s^2\right) + sN_{\mathrm{o}}\left(s^2\right) \\ D(s) &:= D_{\mathrm{e}}\left(s^2\right) + sD_{\mathrm{o}}\left(s^2\right). \end{aligned}$$

- The characteristic polynomial with $s = j\omega$,

$$\begin{aligned} \delta(j\omega) &= \left[-\omega^2 N_{\mathrm{o}}\left(-\omega^2\right) x_1 + N_{\mathrm{e}}\left(-\omega^2\right) x_2 + D_{\mathrm{e}}\left(-\omega^2\right) x_3 - \omega^2 D_{\mathrm{o}}\left(-\omega^2\right)\right] \\ &\quad + j\omega\left[N_{\mathrm{e}}\left(-\omega^2\right) x_1 + N_{\mathrm{o}}\left(-\omega^2\right) x_2 + D_{\mathrm{o}}\left(-\omega^2\right) x_3 + D_{\mathrm{e}}\left(-\omega^2\right)\right]. \end{aligned}$$

- Boundaries of Roots
  - The complex root boundary $\delta(j\omega) = 0, \text{ for } \omega \in (0, +\infty)$
  - The real root boundary $\delta(0) = 0, \quad \delta_{n+1} = 0$

- The complex boundary leads to $\mathrm{Re}[\delta(j\omega)] = 0$ and $\mathrm{Im}[\delta(j\omega)] = 0$.

$$\begin{aligned} -\omega^2 N_o\left(-\omega^2\right) x_1 + N_e\left(-\omega^2\right) x_2 + D_e\left(-\omega^2\right) x_3 - \omega^2 D_o\left(-\omega^2\right) &= 0 \quad (**) \\ \omega\left[N_e\left(-\omega^2\right) x_1 + N_o\left(-\omega^2\right) x_2 + D_o\left(-\omega^2\right) x_3 + D_e\left(-\omega^2\right)\right] &= 0. \quad (*) \end{aligned}$$

- Note that at $\omega = 0$ (*) is trivially satisfied and (**) becomes

$$N_e(0)x_2 + D_e(0)x_3 = 0$$

which coincides with the condition $\delta(0) = 0$.

- The condition $\delta_{n+1} = 0$ translates to $d_n + x_1 n_n = 0$. where d_n, n_n denote the coefficients of s^n in $D(s)$ and $N(s)$ respectively.

- For $\omega > 0$, we have

$$\begin{aligned} -\omega^2 N_\mathrm{o}\left(-\omega^2\right) x_1 + N_\mathrm{e}\left(-\omega^2\right) x_2 + D_\mathrm{e}\left(-\omega^2\right) x_3 - \omega^2 D_\mathrm{o}\left(-\omega^2\right) &= 0 \\ N_\mathrm{e}\left(-\omega^2\right) x_1 + N_\mathrm{o}\left(-\omega^2\right) x_2 + D_\mathrm{o}\left(-\omega^2\right) x_3 + D_\mathrm{e}\left(-\omega^2\right) &= 0. \end{aligned}$$

- Rewrite the above in matrix form as

$$\underbrace{\begin{bmatrix} \omega^2 N_\mathrm{o}\left(-\omega^2\right) & -N_\mathrm{e}\left(-\omega^2\right) \\ N_\mathrm{e}\left(-\omega^2\right) & N_\mathrm{o}\left(-\omega^2\right) \end{bmatrix}}_{A(\omega)} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} D_\mathrm{e}\left(-\omega^2\right) x_3 - \omega^2 D_\mathrm{o}\left(-\omega^2\right) \\ -D_\mathrm{o}\left(-\omega^2\right) x_3 - D_\mathrm{e}\left(-\omega^2\right) \end{bmatrix}.$$

- When $|A(\omega)| \neq 0$ for all $\omega > 0$,

$$|A(\omega)| = \omega^2 N_\mathrm{o}^2\left(-\omega^2\right) + N_\mathrm{e}^2\left(-\omega^2\right) > 0, \qquad \forall \ \omega > 0.$$

- For every x_3, a unique solution x_1 and x_2 at each $\omega > 0$ given by:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{|A(\omega)|} \begin{bmatrix} N_{\rm o}(-\omega^2) & N_{\rm e}(-\omega^2) \\ -N_{\rm e}(-\omega^2) & \omega^2 N_{\rm o}(-\omega^2) \end{bmatrix} \begin{bmatrix} D_{\rm e}(-\omega^2)\, x_3 - \omega^2 D_{\rm o}(-\omega^2) \\ -D_{\rm o}(-\omega^2)\, x_3 - D_{\rm e}(-\omega^2) \end{bmatrix}$$

or

$$\begin{aligned} x_1(\omega) &= \frac{1}{|A(\omega)|}\Big(\left[N_{\rm o}(-\omega^2) D_{\rm e}(-\omega^2) - N_{\rm e}(-\omega^2) D_{\rm o}(-\omega^2)\right] x_3 \\ &\qquad -\omega^2 N_{\rm o}(-\omega^2) D_{\rm o}(-\omega^2) - N_{\rm e}(-\omega^2) D_{\rm e}(-\omega^2)\Big) \\ x_2(\omega) &= \frac{1}{|A(\omega)|}\Big(\left[-N_{\rm e}(-\omega^2) D_{\rm e}(-\omega^2) - \omega^2 N_{\rm o}(-\omega^2) D_{\rm o}(-\omega^2)\right] x_3 \\ &\qquad +\omega^2 N_{\rm e}(-\omega^2) D_{\rm o}(-\omega^2) - \omega^2 N_{\rm o}(-\omega^2) D_{\rm e}(-\omega^2)\Big). \end{aligned}$$

- For a fixed value of x_3, let ω run from 0 to ∞. The above equations trace out a curve in the x_1 - x_2 plane corresponding to the complex root space boundary.

- These curves along with the straight lines from the real root boundary conditions partition the parameter space into a set of open root distribution invariant regions. By sweeping over x_3, we find these regions.

- We now consider the case when $|A(\omega)| = 0$.

- Let

$$|A(\omega)| = \omega^2 N_o^2(-\omega^2) + N_e^2(-\omega^2) = 0, \quad \text{for some } \omega \neq 0. \quad (*)$$

- Since $N_o^2(-\omega^2), N_e^2(-\omega^2) \geq 0$, (*) holds if and only if

$$N_o(\omega^2) = N_e(-\omega^2) = 0.$$

- Recall the matrix equation

$$\underbrace{\begin{bmatrix} \omega^2 N_o(-\omega^2) & -N_e(-\omega^2) \\ N_e(-\omega^2) & N_o(-\omega^2) \end{bmatrix}}_{A(\omega)} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} D_e(-\omega^2)x_3 - \omega^2 D_o(-\omega^2) \\ -D_o(-\omega^2)x_3 - D_e(-\omega^2) \end{bmatrix}.$$

must be 0.

- It means

$$D_{\rm e}\left(-\omega^2\right)x_3 - \omega^2 D_{\rm o}\left(-\omega^2\right) = 0, \quad -D_{\rm o}\left(-\omega^2\right)x_3 - D_{\rm e}\left(-\omega^2\right) = 0$$

- This is equivalent to

$$\omega^2 D_{\rm o}^2\left(-\omega^2\right) + D_{\rm e}^2\left(-\omega^2\right) = 0. \qquad (*)$$

- Since $D_{\rm o}^2\left(\omega^2\right)$, $D_{\rm e}^2\left(-\omega^2\right) \geq 0$, $(*)$ holds iff

$$D_{\rm o}\left(\omega^2\right) = D_{\rm e}\left(-\omega^2\right) = 0.$$

- It follows that $|A(\omega)| = 0$ has a solution for $\omega \neq 0$ iff

$$\begin{aligned} N_{\rm o}\left(\omega^2\right) = N_{\rm e}\left(-\omega^2\right) &= 0 \\ D_{\rm o}\left(\omega^2\right) = D_{\rm e}\left(-\omega^2\right) &= 0 \end{aligned}$$

or $D(s)$ and $N(s)$ have a common factor $s^2 + \omega^2$ and this is ruled out by the assumption of stabilizability of the plant.

Example

- Consider the following 13th order plant

$$P(s) = \frac{s^{10} + 2s^9 + 3s^8 + 4s^7 + 10s^6 + 5s^5 + s^4 - 7s^3 + 4s^2 + s + 23}{s^{13} + 9s^{12} + 40s^{11} + 111s^{10} + 203s^9 + 115s^8 - 203s^7 + 60s^6 + 25s^5 + s^4 - 18s^3 + 21s^2 + 2s + 7}$$

- 1st order controller: $C(s) = \dfrac{x_1 s + x_2}{s + x_3}$

- The characteristic polynomial $\delta(j\omega) = \delta_{\mathrm{r}}(\omega) + j\omega\delta_{\mathrm{i}}(\omega)$ where

$$\begin{aligned}
\delta_{\mathrm{r}}(\omega) = {} & -\omega^{14} + (40 + 9x_3)\,\omega^{12} + (-203 - 2x_1 - x_2 - 111x_3)\,\omega^{10} \\
& + (-203 + 4x_1 + 3x_2 + 115x_3)\,\omega^8 + (-25 - 5x_1 - 10x_2 - 60x_3)\,\omega^6 \\
& + (-18 - 7x_1 + x_2 + x_3)\,\omega^4 + (-2 - x_1 - 4x_2 - 21x_3)\,\omega^2 + (23x_2 + 7x_3)
\end{aligned}$$

$$\begin{aligned}
\delta_{\mathrm{i}}(\omega) = {} & (9 + x_3)\,\omega^{12} + (-111 - x_1 - 40x_3)\,\omega^{10} + (115 + 3x_1 + 2x_2 + 203x_3)\,\omega^8 \\
& + (-60 - 10x_1 - 4x_2 + 203x_3)\,\omega^6 + (1 + x_1 + 5x_2 + 25x_3)\,\omega^4 \\
& + (-21 - 4x_1 + 7x_2 + 18x_3)\,\omega^2 + (7 + 23x_1 + x_2 + 2x_3)\,.
\end{aligned}$$

- For $\omega = 0$, $23x_2 + 7x_3 = 0$,
- For $\omega > 0$,

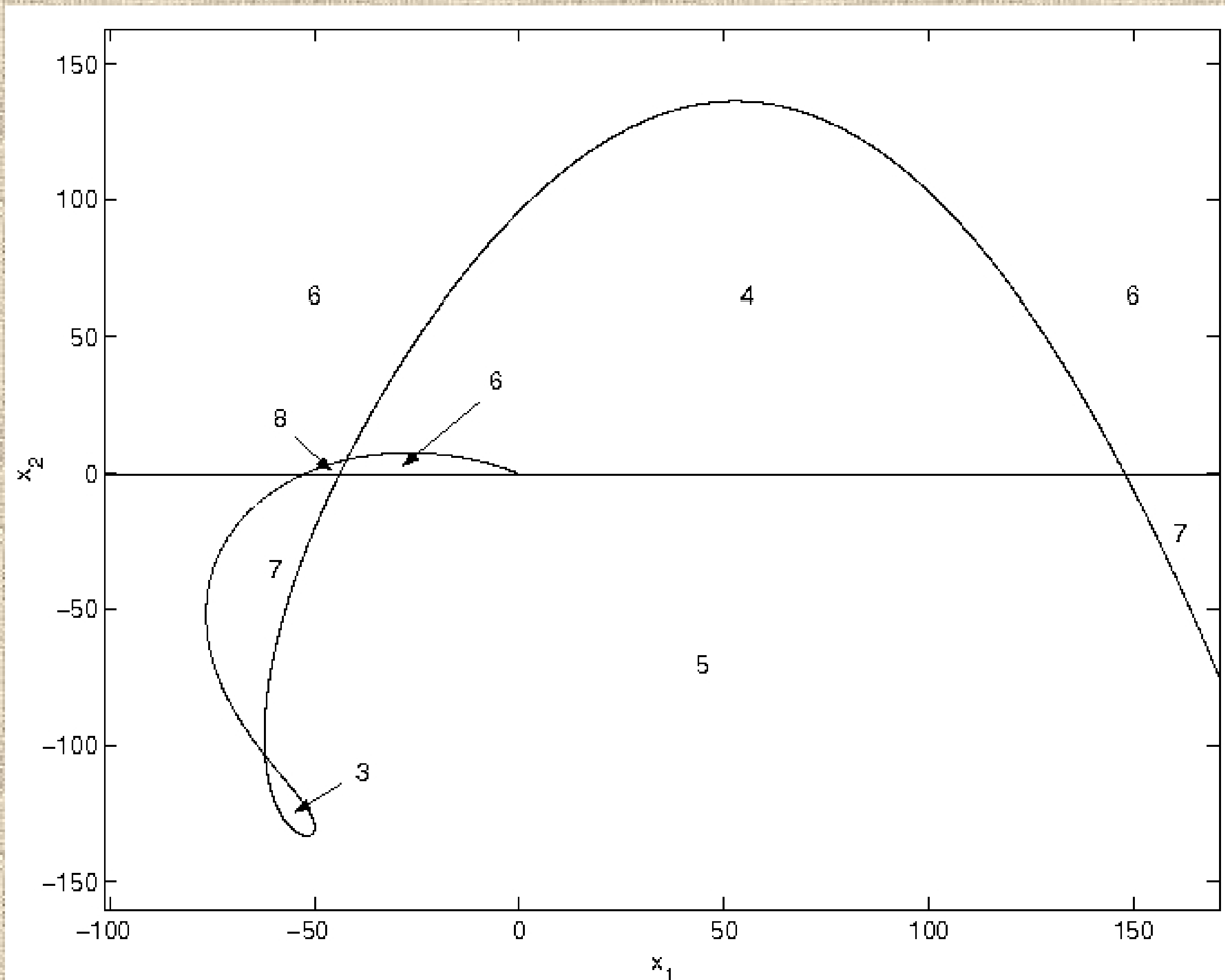
$$
\begin{aligned}
x_1(\omega) &= \frac{-9\omega^{22} + 120\omega^{20} - 181\omega^{18} - 452\omega^{16} + 1429\omega^{14} - 1738\omega^{12} + 3355\omega^{10} - 2931\omega^8 + 1586\omega^6 - 142\omega^4 + 504\omega^2 - 161}{|A(\omega)|} \\
&\quad + \frac{9\omega^{22} - 120\omega^{20} + 280\omega^{18} - 805\omega^{16} + 406\omega^{14} - 1341\omega^{12} + 3501\omega^{10} - 3319\omega^8 + 1289\omega^6 - 225\omega^4 + 539\omega^2 - 154}{|A(\omega)|} \cdot x_3
\end{aligned}
$$

$$
\begin{aligned}
x_2(\omega) &= \frac{-9\omega^{24} + 120\omega^{22} - 280\omega^{20} + 805\omega^{18} - 406\omega^{16} + 1341\omega^{14} - 3501\omega^{12} + 3319\omega^{10} - 1289\omega^8 + 225\omega^6 - 539\omega^4 + 154\omega^2}{|A(\omega)|} \\
&\quad + \frac{-27\omega^{22} + 396\omega^{20} - 1257\omega^{18} + 2596\omega^{16} - 1527\omega^{14} + 1042\omega^{12} - 2951\omega^{10} + 3303\omega^8 - 1364\omega^6 + 86\omega^4 - 518\omega^2 + 161}{|A(\omega)|} \cdot x_3
\end{aligned}
$$

where

$$
\begin{aligned}
|A(\omega)| &= \omega^{20} - 2\omega^{18} + 13\omega^{16} - 26\omega^{14} + 102\omega^{12} - 117\omega^{10} + 281\omega^8 \\
&\quad -409\omega^6 + 76\omega^4 - 183\omega^2 + 529.
\end{aligned}
$$

For a fixed x_3



EXAMPLE

- Plant:

$$P(s) := \frac{3s^7 + 99s^6 + 1320s^5 + 9255s^4 + 37287s^3 + 88656s^2 + 120420s + 75600}{s^8 + 18s^7 + 131s^6 + 625s^5 + 2017s^4 + 4753s^3 + 7896s^2 + 8919s + 5670}$$

- 1st Order Controller: $C(s) = \dfrac{x_1 s + x_2}{s + x_3}$

- Then we have

$$\begin{aligned} N_e(s) &= 99s^6 + 9255s^4 + 88656s^2 + 75600 \\ N_o(s) &= 3s^6 + 1320s^4 + 37287s^2 + 120420 \\ D_e(s) &= s^8 + 131s^6 + 2017s^4 + 7896s^2 + 5670 \\ D_o(s) &= 18s^6 + 625s^4 + 4753s^2 + 8919 \end{aligned}$$

- The characteristic polynomial $\delta(j\omega) = \delta_r(\omega) + j\omega\delta_i(\omega)$

$$\begin{aligned} \delta_r(\omega) &= (18 + 3x_1 + x_3)\,\omega^8 + (-625 - 1320x_1 - 99x_2 - 131x_3)\,\omega^6 + (4753 + 37287x_1 + 9255x_2 + 2017x_3)\,\omega^4 \\ &\quad + (-8919 - 120420x_1 - 88656x_2 - 7896x_3)\,\omega^2 + 75600x_2 + 5670x_3 \\ \delta_i(\omega) &= \omega^8 + (-131 - 99x_1 - 3x_2 - 18x_3)\,\omega^6 + (2017 + 9255x_1 + 1320x_2 + 625x_3)\,\omega^4 \\ &\quad + (-7896 - 88656x_1 - 37287x_2 - 4753x_3)\,\omega^2 + (5670 + 75600x_1 + 120420x_2 + 8919x_3)\,. \end{aligned}$$

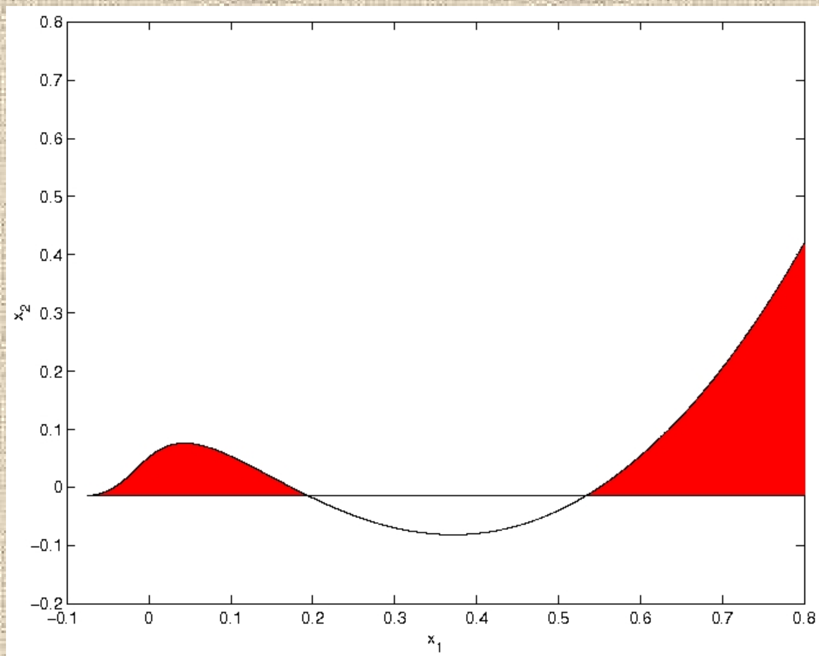
- For $\omega = 0, \quad 75600x_2 + 5670x_3 = 0,$

- For $\omega > 0,$

$$
\begin{aligned}
x_1(\omega) &= \frac{45\omega^{14} + 3411\omega^{12} + -9681\omega^{10} + 634077\omega^8 - 1899129\omega^6 - 69813\omega^4 + 25591140\omega^2 - 428652000}{|A(\omega)|} \\
&+ \frac{-3\omega^{14} - 69\omega^{12} + 12207\omega^{10} - 159585\omega^8 + 220167\omega^6 - 6387621\omega^4 - 12203946\omega^2 + 8505000}{|A(\omega)|} \cdot x_3 \\
x_2(\omega) &= \frac{\omega^{16} + 69\omega^{14} - 12207\omega^{12} + 159585\omega^{10} - 220167\omega^8 + 6387621\omega^6 + 12203946\omega^4 - 8505000\omega^2}{|A(\omega)|} \\
&+ \frac{-153\omega^{14} + 47859\omega^{12} - 3011169\omega^{10} + 62911227\omega^8 - 526622253\omega^6 + 1809907839\omega^4 - 2173643100\omega^2 + 428652000}{|A(\omega)|} \cdot x_3.
\end{aligned}
$$

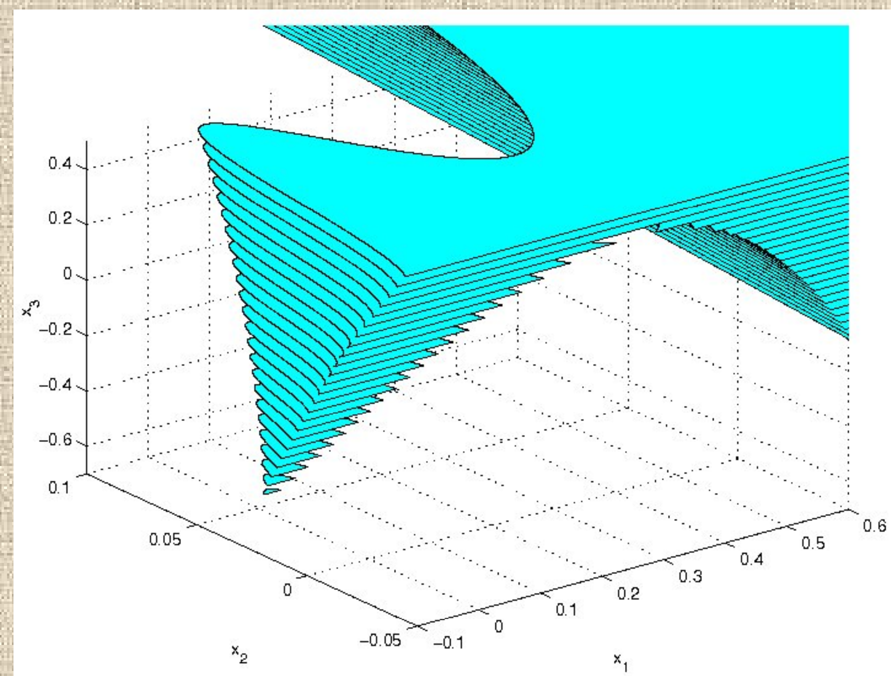
where

$$
\begin{aligned}
|A(\omega)| &= 9\omega^{14} + 1881\omega^{12} + 133632\omega^{10} + 4048713\omega^8 + 52237809\omega^6 \\
&\quad +279041256\omega^4 + 1096189200\omega^2 + 5715360000.
\end{aligned}
$$



Stability region for $x_3 = 0.2$

Stability region for $-0.7 \leq x_3 \leq 0.5$



ROBUST STABILIZATION BY FOCs

- Robust stabilization problem considers the plant $P(s) = \frac{N(s)}{D(s)}$ where $P(s)$ is an interval plant.

- Consider a polynomial family of the form $\delta(s) = F_1(s)D(s) + F_2(s)N(s)$ where $N(s)$ and $D(s)$ are interval polynomials and

$$F_i(s) = s^{t_i}(a_i s + b_i)U_i(s)Q_i(s), \qquad i = 1, 2.$$

- $t_i \geq 0$ is an arbitrary integer, a_i and b_i are arbitrary real, $U_i(s)$ is an anti-Hurwitz, and $Q_i(s)$ is an even or odd polynomial.

- Let $\mathbf{\Delta}_{\mathrm{v}}(s) := \{\delta_{\mathrm{v}}(s) \ : \ F_1(s)D_i(s) + F_2(s)N_j(s), \quad i, j = 1, 2, 3, 4\}$

  where $D_i(s)$ and $N_j(s)$ are the Kharitonov polynomials of $D(s)$ and $N(s)$, respectively.

Generalized Kharitonov Theorem

For robust stability of a family $\delta(s)$, it is enough that $\underline{F}(s) = (F_1(s), F_2(s))$ stabilizes the finite set of vertex polynomials $\mathbf{\Delta}_{\mathrm{v}}(s)$.

- For the problem of robust stabilization with 1st order controllers, we consider

$$F_1(s) = s + x_3 \qquad \text{and} \qquad F_2(s) = x_1 s + x_2.$$

- For the given interval plant

$$\mathbf{P}(s) := \left\{ P(s) = \frac{N(s)}{D(s)} \;:\; N(s) \in \mathbf{N}(s), \;\; D(s) \in \mathbf{D}(s) \right\}$$

- Let its vertices be

$$\mathbf{P}_{\mathrm{v}}(s) := \left\{ \frac{N_i(s)}{D_j(s)} \;:\; N_i(s) \in \mathcal{K}_N(s), \;\; D_j(s) \in \mathcal{K}_D(s) \right\}$$

where $\mathcal{K}_N(s)$ and $\mathcal{K}_D(s)$ are the set of Kharitonov polynomials of $\mathbf{N}(s)$ and $\mathbf{D}(s)$, respectively.

Robust Stabilization with FOC

Let R_k be controller parameter space regions that consist of all first order stabilizing controllers for the k^{th} vertex system $P_k(s)$. Then every first order stabilizing controller that robustly stabilizes the interval system $\mathbf{P}(s)$ is given by

$$\mathcal{R} := \cap_k \mathrm{R}_k.$$

Example

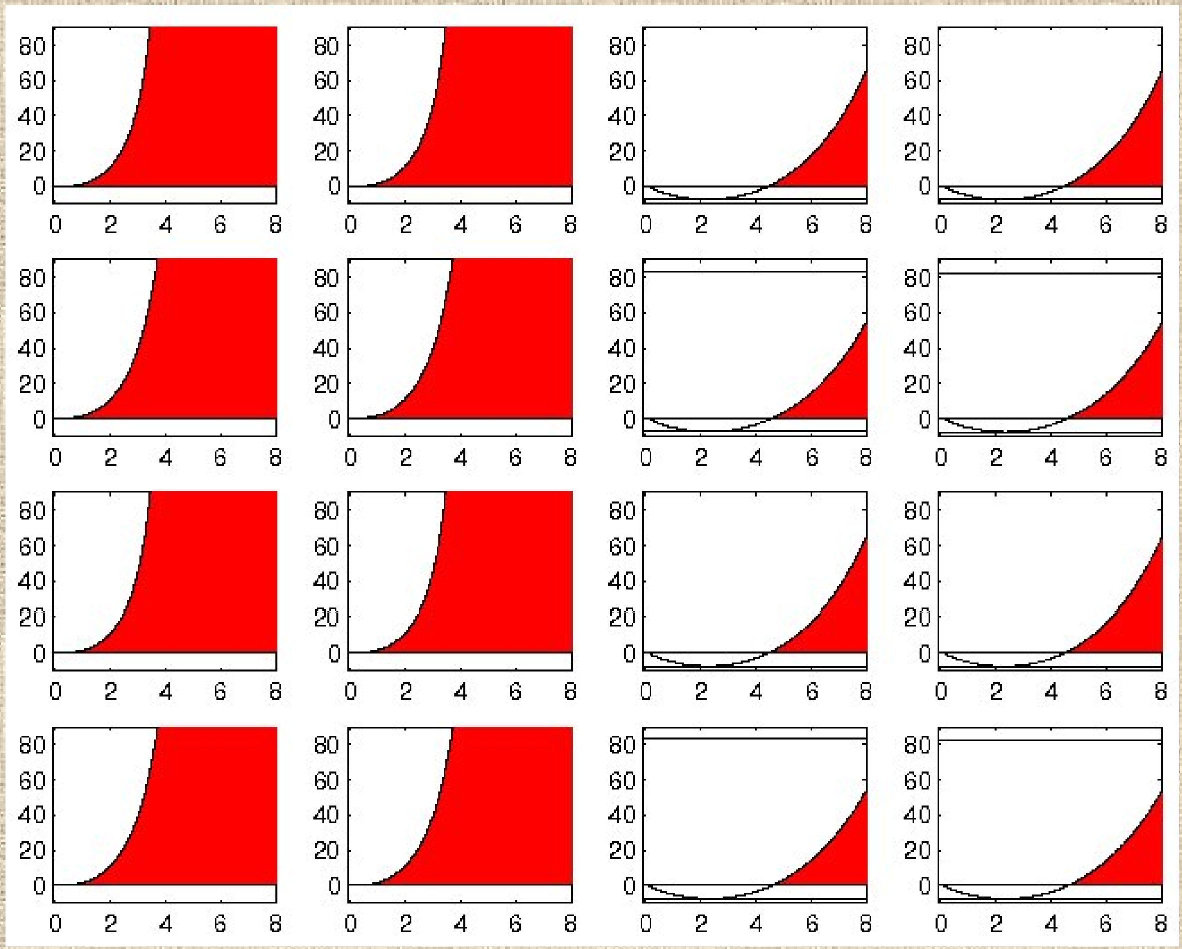
- Plant
$$\mathbf{P}(s) = \left\{\frac{N(s)}{D(s)}\right\} = \left\{\frac{n_7 s^7 + n_6 s^6 + n_5 s^5 + n_4 s^4 + n_3 s^3 + n_2 s^2 + n_1 s + n_0}{d_8 s^8 + d_7 s^7 + d_6 s^6 + d_5 s^5 + d_4 s^4 + d_3 s^3 + d_2 s^2 + d_1 s + d_0}\right\}$$

- The upper and lower bounds of the coefficients of interval polynomials $\mathbf{D}(s)$ and $\mathbf{N}(s)$ are

$$\begin{aligned} D^+ &= \{2,\ 18.1,\ 131.1,\ 625.1,\ 2017.1,\ 4753.1,\ 7896.1,\ 8919.1,\ 5670.1\} \\ D^- &= \{1,\ 17.9,\ 130.9,\ 624.9,\ 2016.9,\ 4752.9,\ 7895.9,\ 8918.9,\ 5669.9\} \\ N^+ &= \{3.1,\ 99.1,\ 1320.1,\ 9255.1,\ 37287.1,\ 88656.1,\ 120420.1,\ 75600.1\} \\ N^- &= \{2.9,\ 98.9,\ 1319.9,\ 9254.9,\ 37286.9,\ 88655.9,\ 120419.9,\ 75599.9\} \end{aligned}$$

- We now construct the 16 vertex systems:

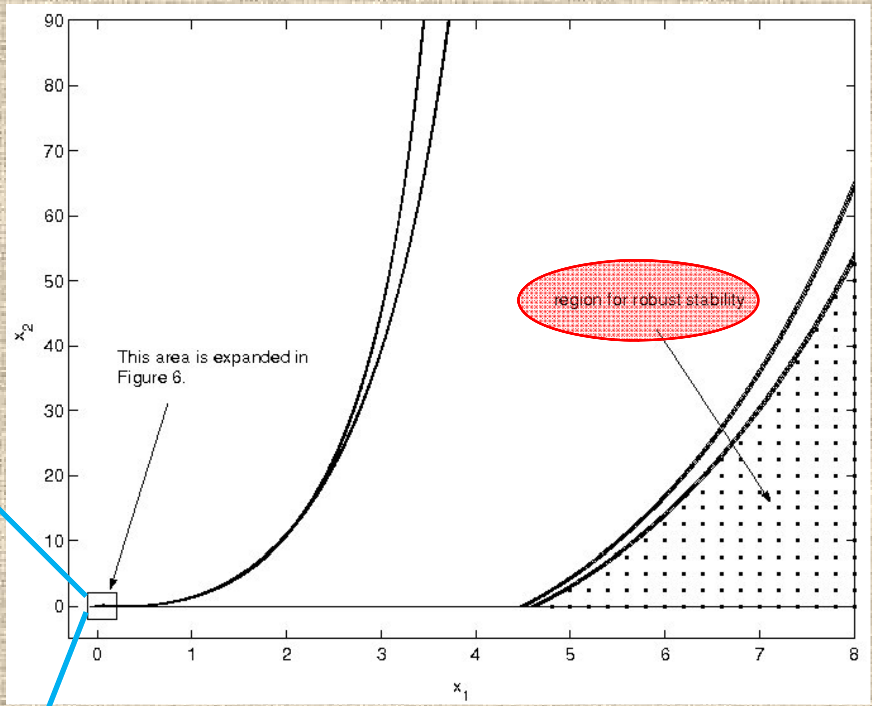
$$\mathbf{P}_{\mathrm{v}}(s) = \left\{P_l(s) \ : \ \frac{N_i(s)}{D_j(s)},\quad N_i(s) \in \mathcal{K}_N(s),\ D_j(s) \in \mathcal{K}_D(s)\right\}$$

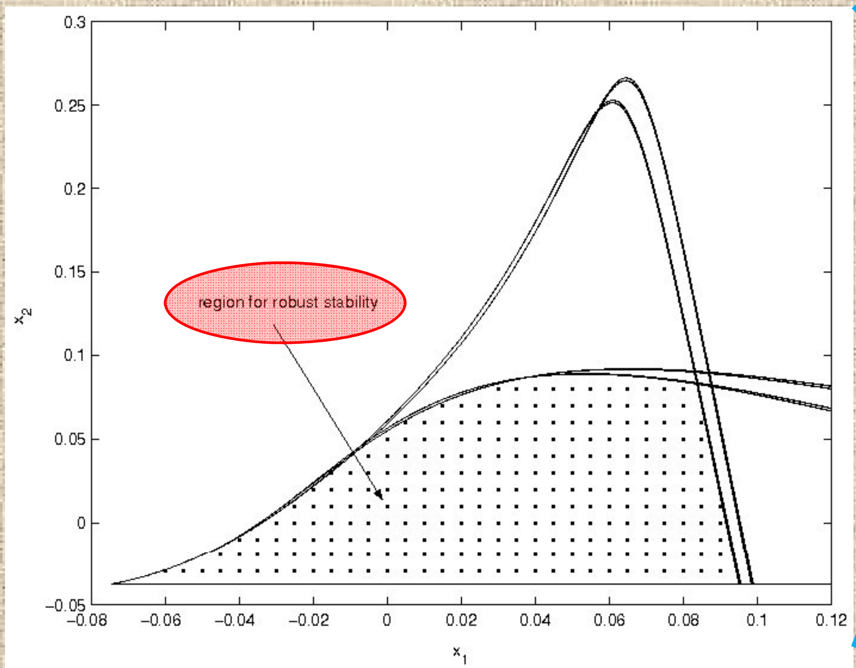


Stabilizing regions for all 16 vertex systems

- To observe the intersection of these 16 regions, we consider

Region for robust stability when $x_3 = 0.5$





Area magnified

H_∞ DESIGN WITH FOC

- The gain margin and phase margin specifications can be imposed by replacing the plant by $KP(s)$ and $e^{(-j\theta)}P(s)$ respectively.
- Consider the standard feedback configuration
- The plant is assumed to be SISO, LTI, proper, and coprime.
- For a given closed loop transfer function $T(s)$ and positive scalar γ, define the H_∞ design criteria to be:

$$\|W(s)T(s)\|_\infty < \gamma \quad \text{where} \quad W(s) = \frac{W_n(s)}{W_d(s)}$$

is a stable, coprime, frequency-dependent weighting function.

- Consider the complementary sensitivity function

$$T(s) = \frac{(x_1 s + x_2)\, N(s)}{(s + x_3)\, D(s) + (x_1 s + x_2)\, N(s)}.$$

- The objective is to determine the region in the parameter space for which the closed loop system is stable and the above defined H_∞ optimization criteria is satisfied.

- Specifically, the objective is to determine values of x_1, x_2, and x_3 (if any) for the controller $C(s)$, such that

$$\left\| \frac{W_n(s)}{W_d(s)} \left[\frac{(x_1 s + x_2) N(s)}{(s + x_3) D(s) + (x_1 s + x_2) N(s)} \right] \right\|_\infty < \gamma$$

and the closed loop characteristic polynomial

$$\delta(s) := (s + x_3) D(s) + (x_1 s + x_2) N(s) \ \in \mathcal{H}$$

LEMMA

Let
$$F(s) = \frac{N_F(s)}{D_F(s)}$$
be a stable and proper rational function, where $N_F(s)$ and $D_F(s)$ are polynomials with $deg[D_F(s)] =: q$ and n_q, d_q denote the q^{th} degree coefficients of N_F, D_F. Define
$$\phi(s) := D_F(s) + \frac{1}{\gamma} e^{j\theta} N_F(s).$$
Then for a given $\gamma > 0$, $\|F(s)\|_\infty < \gamma$ if and only if

1. $|n_q| < \gamma |d_q|$
2. $\phi(s)$ is Hurwitz for all θ in $[0, 2\pi)$,

where n_q and d_q are the coefficients of s^q in $N_F(s)$ and $D_F(s)$, respectively.

- We now apply this result to our problem.

- Set
$$\begin{aligned} N_F(s) &= W_n(s)\,(x_1 s + x_2)\,N(s) \\ D_F(s) &= W_d(s)\Big[(s + x_3)\,D(s) + (x_1 s + x_2)\,N(s)\Big]. \end{aligned}$$

- Let
$$\begin{aligned} W_n(s) &= a_m s^m + a_{m-1} s^{m-1} + \cdots + a_1 s + a_0 \\ W_d(s) &= b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0 \\ N(s) &= \alpha_n s^n + \alpha_{n-1} s^{n-1} + \cdots + \alpha_1 s + \alpha_0 \\ D(s) &= \beta_n s^n + \beta_{n-1} s^{n-1} + \cdots + \beta_1 s + \beta_0 \end{aligned}$$
where b_m and $\beta_n \neq 0$

- Then $n_q = \alpha_n a_m x_1$ and $d_q = b_m\,(\beta_n + \alpha_n x_1)$.

Problem Formulation

Determine the region in the controller parameter space such that

- $\delta(s) := (s + x_3)D(s) + (x_1 s + x_2)N(s) \ \in \mathcal{H}$
- $|n_q| < \gamma |d_q|$ and
- $\phi(s) := D_F(s) + \frac{1}{\gamma} e^{j\theta} N_F(s) \quad \in \mathcal{H}$ for all $\theta \in [0.2\pi)$.

- The problem of determining the controller to satisfy the above simultaneous stabilization conditions can be solved as before by the D-decomposition technique with fixed x_3.

Example

- Plant and controller: $G(s) = \dfrac{s-1}{s^2 + 0.8s - 0.2}$ and $C(s) = \dfrac{x_1 s + x_2}{s + x_3}$

- The closed loop transfer function is

$$T(s) = \frac{(x_1 s + x_2)(s-1)}{(s+x_3)(s^2+0.8s-0.2) + (x_1 s + x_2)(s-1)}.$$

- We arbitrarily choose $x_3 = 2.5$, $\gamma = 1$, and the weighting function to be the high pass transfer function

$$W(s) = \frac{s+0.1}{s+1}.$$

- Determine the set of all stabilizing controllers, $C(s)$ such that $\|W(s)T(s)\|_\infty < 1$ for $x_3 = 2.5$.

- First, determine the stability region
- Note that the plant is strictly proper so that $|n_\alpha| < \gamma |d_\alpha|$ for all x_1, x_2, and x_3.
- Determine the region for which $\phi(s)$ is Hurwitz for all $\theta \in [0, 2\pi)$ where

$$\begin{aligned}\phi(s) &= (s+1)\Big[(s+x_3)\left(s^2+0.8s-0.2\right)+(x_1 s+x_2)(s-1)\Big] \\ &\quad +e^{j\theta}(s+0.1)(x_1 s+x_2)(s-1).\end{aligned}$$

- Using the substitution, $e^{j\theta} = \alpha + j\beta$,

$$\begin{aligned}\phi(j\omega) &= \omega^4+\beta x_1\omega^3+(-0.6+0.9\alpha x_1-x_2-\alpha x_2-1.8x_3)+(0.1\beta x_1+0.9\beta x_2)\,\omega \\ &\qquad -(1+0.1\alpha)x_2-0.2x_3 \\ &\quad +j\left[(-1.8-x_1-\alpha x_1-x_3)\,\omega^3+(0.9\beta x_1-\beta x_2)\,\omega^2\right. \\ &\qquad \left.+(-0.2-x_1-0.1\alpha x_1-0.9\alpha x_2+0.6x_3)\,\omega-0.1\beta x_2\right].\end{aligned}$$

- First, we consider the real root boundaries. The real root boundary for the origin is obtained by setting $\phi(0) = 0$:

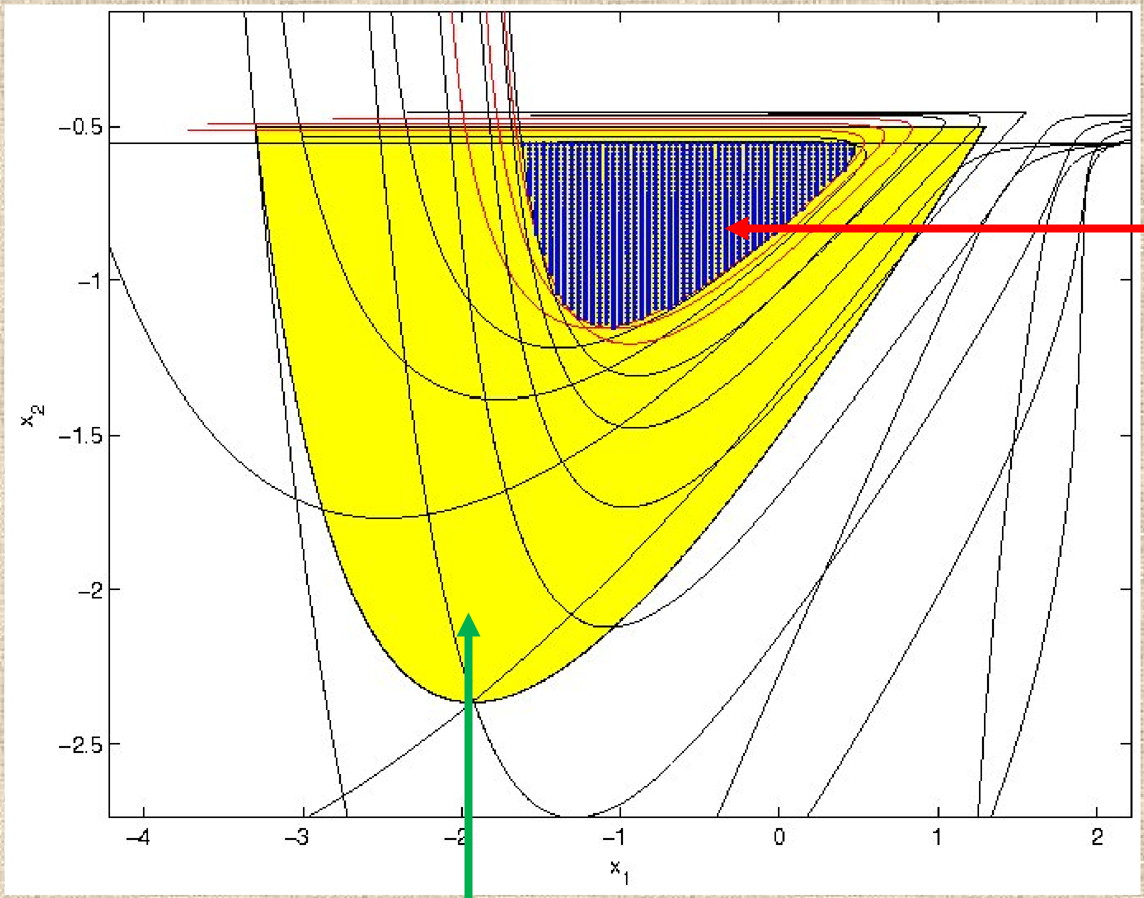
$$-(1 + 0.1\alpha)x_2 - 0.2x_3 = 0, \qquad -0.1\beta x_2 = 0$$

⇓

$$x_2 = \left(\frac{-0.2}{1 + 0.1\alpha}\right) x_3$$

- Since the plant is strictly proper, there is not a real root boundary at infinity.

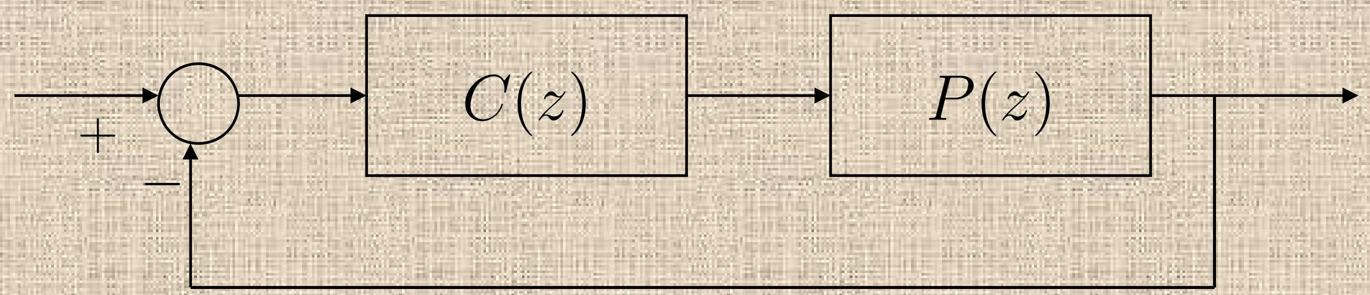
- The complex root boundary is characterized by $\phi(j\omega) = 0$.



stability region satisfying H_∞ constaint

stability region

FO DISCRETE-TIME CONTROLLERS



- Plant and controllers:

$$P(z) = \frac{N(z)}{D(z)}, \qquad C(z) = \frac{x_1 z + x_2}{z + x_3} =: \frac{N_c(z)}{D_c(z)}$$

- The characteristic polynomial is

$$\begin{aligned}\delta(z) &= D_c(z)D(z) + N_c(z)N(z) \\ &= (z + x_3)\,D(z) + (x_1 z + x_2)\,N(z)\end{aligned}$$

- We now determine the image of a polynomial

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_2 z^2 + a_1 z + a_0$$

with real coefficients, evaluated on the unit circle.

- Setting

$$u := -\cos\theta,$$

- we have $$z^k = e^{jk\theta} = \cos k\theta + j \sin k\theta.$$

- Using Tchebyshev representations, we have

$$P\left(e^{j\theta}\right) = R_P(u) + j\sqrt{1-u^2}\,T_P(u)$$

where
$$\begin{aligned} R_P(u) &= a_n c_n(u) + a_{n-1} c_{n-1}(u) + \cdots + a_1 c_1(u) + a_0 \\ T_P(u) &= a_n s_n(u) + a_{n-1} s_{n-1}(u) + \cdots + a_1 s_1(u). \end{aligned}$$

are real polynomials of degree n and $n-1$ respectively.

- Consider the characteristic polynomial

$$\delta(z) = (z + x_3)\, D(z) + (x_1 z + x_2)\, N(z)$$

- Then we write

$$D(e^{j\theta}) := R_D(u) + j\sqrt{1-u^2}T_D(u), \qquad N(e^{j\theta}) := R_N(u) + j\sqrt{1-u^2}T_N(u)$$

$$\begin{aligned} e^{j\theta} + x_3 &= -u + j\sqrt{1-u^2} + x_3 \\ x_1 e^{j\theta} + x_2 &= -x_1 u + j\sqrt{1-u^2}x_1 + x_2. \end{aligned}$$

- The characteristic polynomial evaluated on the unit circle then becomes

$$\Pi(u) = \Pi_r(u) + j\Pi_i(u)$$

where

$$\begin{aligned} \Pi_r(u) &= R_D(u)\,(x_3 - u) - T_D(u)\,(1-u^2) \\ &\qquad + R_N(u)\,(x_2 - u x_1) - (1-u^2)\, x_1 T_N(u) \\ \Pi_i(u) &= \sqrt{1-u^2}\,[R_D(u) + T_D(u)\,(x_3 - u) \\ &\qquad + x_1 R_N(u) + T_N(u)\,(x_2 - u x_1)] \end{aligned}$$

- The stability bounday for complex roots is given by seting

$$\Pi(u) = 0, \qquad u \in (-1, 1)$$

- The stability boundaries for real roots are given by

$$\Pi(-1) = 0, \qquad \Pi(1) = 0.$$

- The complex root boundary from the Boundary Crossing Conditions is given by

$$\Pi_r(u) = 0, \ (*) \qquad \Pi_i(u) = 0 \quad (**)$$

- Note that at $u = 1$ and $u = -1$, (**) is trivially satisfied, that is holds for all x_1, x_2, and x_3.

- At $u = 1$, (*) becomes

$$R_D(1)\,(x_3 - 1) + R_N(1)\,(x_2 - x_1) = 0$$

which for a given x_3 is a straight line in the $x_1 - x_2$ plane.

- Similarly, at $u = -1$,

$$R_D(-1)\,(x_3 + 1) + R_N(-1)\,(x_2 + x_1) = 0$$

which for fixed x_3 is a straight line.

- For $-1 < u < 1$,

$$\Pi_r(u) = 0, \qquad \Pi_i(u) = 0$$

$$\begin{bmatrix} (u^2-1)\,T_N(u) - uR_N(u) & R_N(u) \\ R_N(u) - uT_N(u) & T_N(u) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} (u - x_3)\,R_D(u) + (1-u^2)\,T_D(u) \\ (u - x_3)\,T_D(u) - R_D(u) \end{bmatrix}$$

and

$$|A(u)| = T_N^2(u)\left(u^2 - 1\right) - R_N^2(u).$$

- A unique solution x_1 and x_2 at each $u \in (-1, 1)$ is obtained from

$$x_1(u) = \frac{Y(u)\,(u - x_3) + (1 - u^2)\,T_D(u)T_N(u) + R_D(u)R_N(u)}{|A(u)|}$$
$$x_2(u) = \frac{Y(u)\,(1 - ux_3) + x_3\,[R_N(u)R_D(u) + (1 - u^2)\,T_N(u)T_D(u)]}{|A(u)|}$$

where
$$Y(u) = R_D(u)T_N(u) - R_N(u)T_D(u).$$

- The above two equations trace out a curve in the x_1 - x_2 plane representing the complex root space boundary, for fixed x_3, as u runs from -1 to +1.

- This curve along with the lines and partition the controller parameter space into regions with a fixed number of outside the unit circle roots.

- By sweeping over x_3, we can identify the three dimensional stability region for a given plant, if one exists.

Example

- Consider the following 6th order discrete-time plant

$$P(z) = \frac{24z^5 + 72z^4 + 19z^3 + 81z^2 + 84z + 95}{76z^6 + 42z^5 + 56z^4 + 59z^3 + 24z^2 + z + 15}$$

- A controller

$$C(z) = \frac{x_1 z + x_2}{z + x_3}$$

- Choosing $x_3 = 0.75$, the lines corresponding to $u = -1$ and $u = 1$ are

$$u = -1: \quad x_2 = -x_1 + (1 + x_3)\left[\frac{273}{375}\right] = -x_1 - 1.274$$

$$u = 1: \quad x_2 = x_1 + (1 - x_3)\left[\frac{69}{121}\right] = x_1 + 0.14256.$$

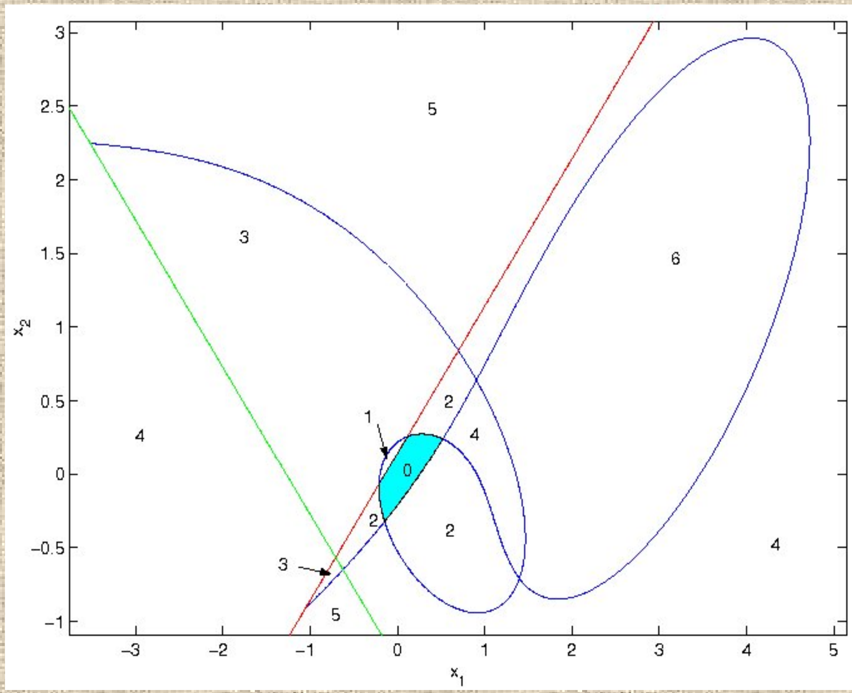
- For $-1 < u < 1$, we have the curve given parametrically by

$$\begin{aligned} x_1(u) &= \frac{462080u^6 - 505248u^5 - 217368u^4 + 303488u^3 + 13524u^2 - 38088u - 3014.25}{|A(u)|} \\ x_2(u) &= \frac{222400u^5 - 233616u^4 - 54554u^3 + 86446u^2 - 3246u - 4143.5}{|A(u)|} \end{aligned}$$

where

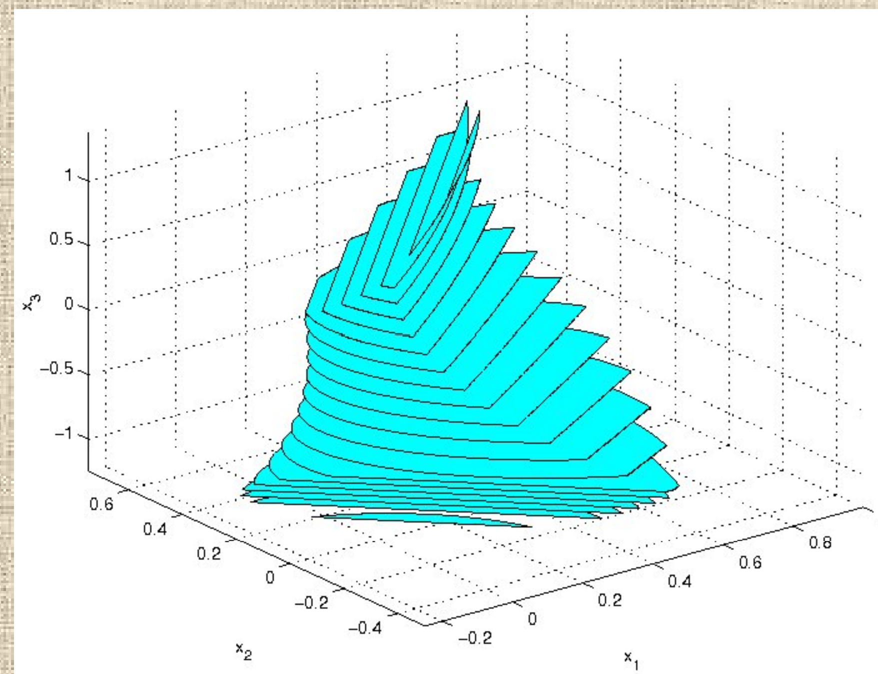
$$|A(u)| = 72960u^5 - 141696u^4 - 12824u^3 + 79380u^2 + 2856u - 15317.$$

- Figure next illustrates how the curve and lines partition the controller parameter space for a fixed $x_3 = 0.75$. Despite the simple controller structure, the behavior of the curve is extremely complicated.



stability region with $x_3 = 0.75$

stability region for $-1.25 \leq x_3 \leq 1.375$



- The figure shows that the region gets smaller as x_3 approaches $+1.375$ and -1.25.

Extension to Design 1st Order Controllers Achieving Maximum Delay Tolerance

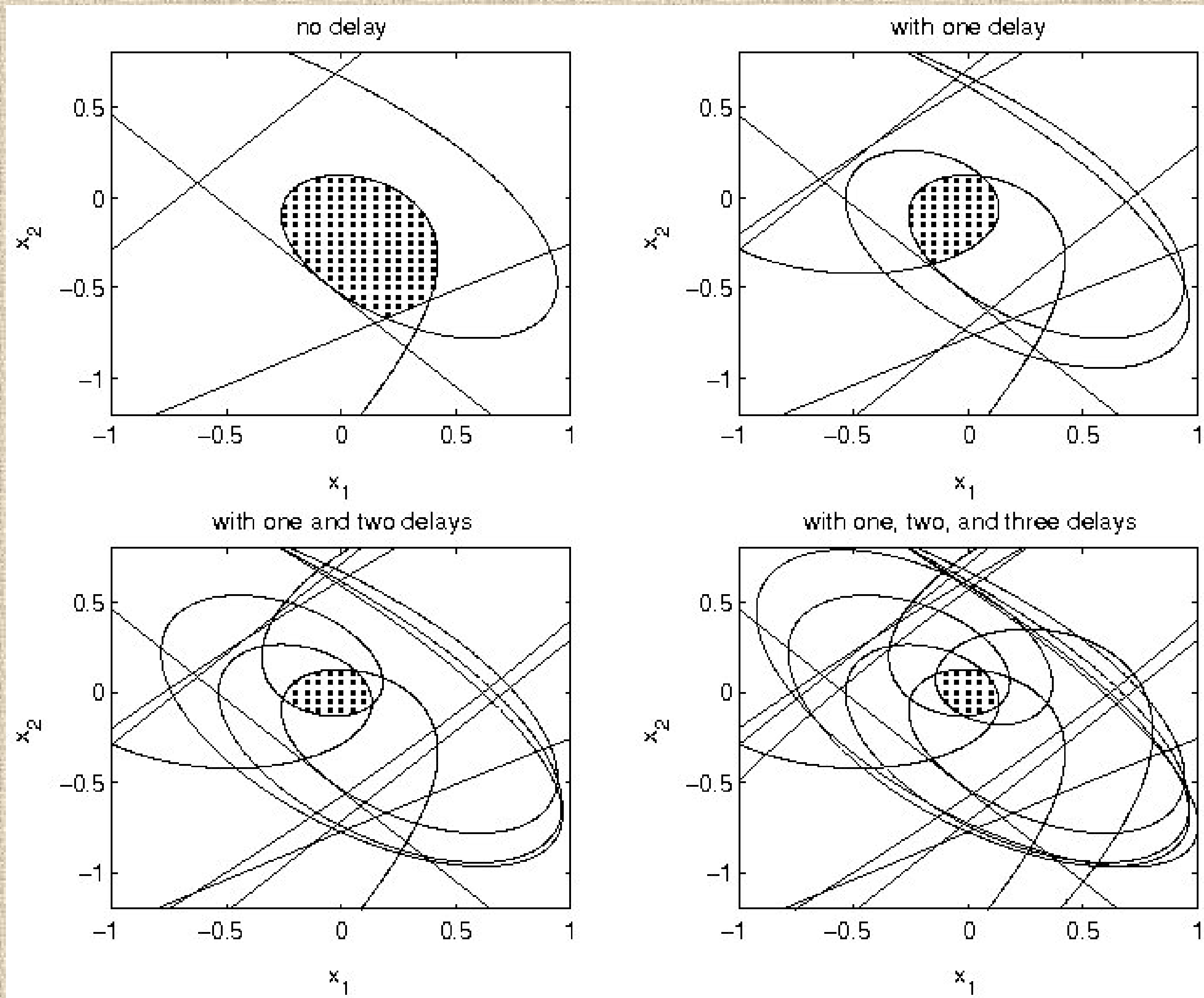
- Delay tolerance can be accomplished by solving the problem of simultaneously stabilizing the systems

$$P(z), \quad z^{-1}P(z), \quad z^{-2}P(z), \quad \cdots, \quad z^{-q}P(z).$$

Example

- Consider the plant given in the previous example with $x_3 = -0.25$.
- As shown, the stability region shrinks as q increases.

First Order Discrete-time Controllers



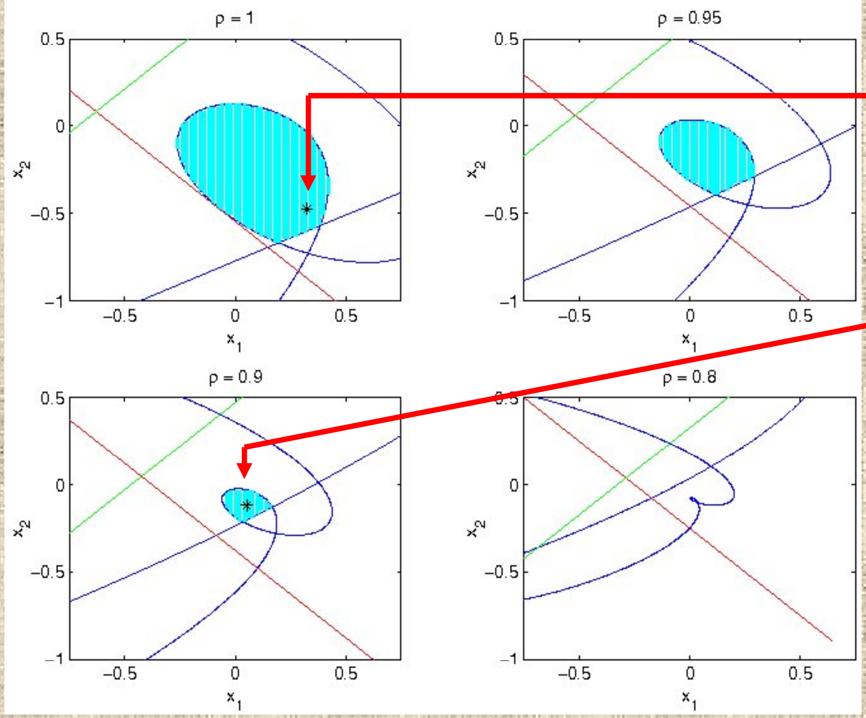
Maximally Deadbeat Design

- This problem can be solved by obtaining the Tchebyshev representation of the characteristic equation on a circle of radius ρ.
- By reducing the value of ρ from 1 we can determine the value of $\rho = \rho^*$ for which the set of stabilizing first order controllers just becomes empty.

Example

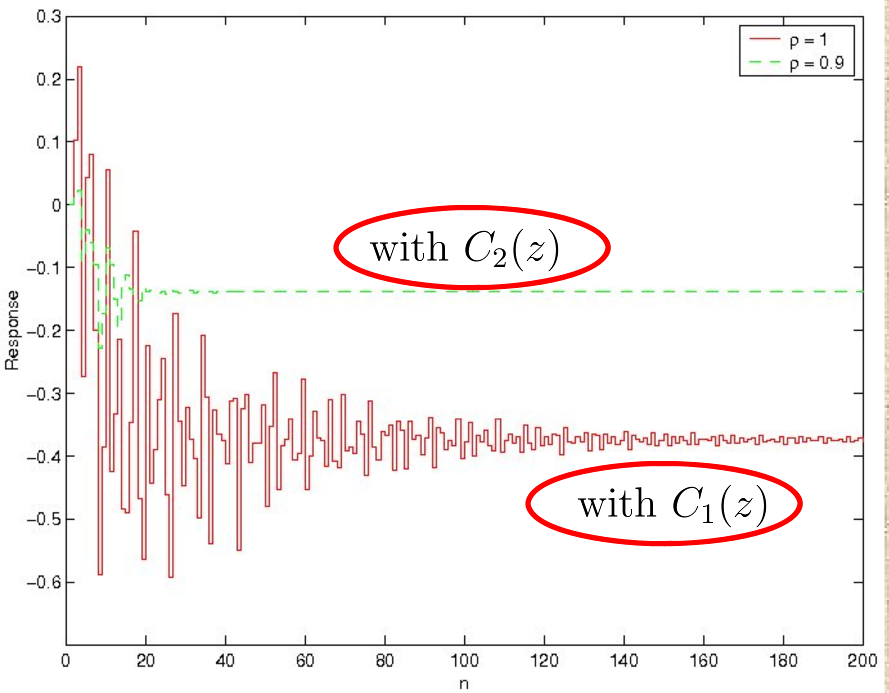
- Consider the plant - controller pair of the previous example.

- The "stabilizing region" for $x_3 = -0.25$ and various values of ρ are shown in the figure. The stabilizing region vanishes for $\rho = 0.8$.



A point selected is (0.3249,-0.4737)
Controller $C_1(z)$

A point selected is (0.0518,-0.1184)
Controller $C_2(z)$



- Step responses illustrate the approximate deadbeat property