First Order Controllers for LTI Systems
First Order Controllers for LTI Systems

- Plant and Controller: $P(s) := \frac{N(s)}{D(s)}$ and $C(s) := \frac{x_1 s + x_2}{s + x_3}$

- Using the standard even-odd decomposition of polynomials
  
  $N(s) := N_e(s^2) + sN_o(s^2)$
  $D(s) := D_e(s^2) + sD_o(s^2)$.

- The characteristic polynomial with $s = j\omega$,
  
  $\delta(j\omega) = [-\omega^2 N_o(-\omega^2)x_1 + N_e(-\omega^2)x_2 + D_e(-\omega^2)x_3 - \omega^2 D_o(-\omega^2)]$
  $+ j\omega [N_e(-\omega^2)x_1 + N_o(-\omega^2)x_2 + D_o(-\omega^2)x_3 + D_e(-\omega^2)]$. 
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- **Boundaries of Roots**
  - The complex root boundary
    \[ \delta(j\omega) = 0, \quad \text{for } \omega \in (0, +\infty) \]
  - The real root boundary
    \[ \delta(0) = 0, \quad \delta_{n+1} = 0 \]

- The complex boundary leads to \( \text{Re}[\delta(j\omega)] = 0 \) and \( \text{Im}[\delta(j\omega)] = 0 \).

\[
\begin{align*}
-\omega^2 N_o (-\omega^2) x_1 + N_e (-\omega^2) x_2 + D_e (-\omega^2) x_3 - \omega^2 D_o (-\omega^2) &= 0 \quad (***) \\
\omega \left[ N_e (-\omega^2) x_1 + N_o (-\omega^2) x_2 + D_o (-\omega^2) x_3 + D_e (-\omega^2) \right] &= 0. \quad (*)
\end{align*}
\]

- Note that at \( \omega = 0 \) \((*)\) is trivially satisfied and \((***)\) becomes

\[ N_e(0)x_2 + D_e(0)x_3 = 0 \]

which coincides with the condition \( \delta(0) = 0 \).
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- The condition $\delta_{n+1} = 0$ translates to $d_n + x_1 n_n = 0$, where $d_n, n_n$ denote the coefficients of $s^n$ in $D(s)$ and $N(s)$ respectively.

- For $\omega > 0$, we have
  
  \[-\omega^2 N_o (-\omega^2) x_1 + N_e (-\omega^2) x_2 + D_e (-\omega^2) x_3 - \omega^2 D_o (-\omega^2) = 0\]
  
  \[N_e (-\omega^2) x_1 + N_o (-\omega^2) x_2 + D_o (-\omega^2) x_3 + D_e (-\omega^2) = 0.\]

- Rewrite the above in matrix form as
  
  \[
  \begin{bmatrix}
  \omega^2 N_o (-\omega^2) & -N_e (-\omega^2) \\
  N_e (-\omega^2) & N_o (-\omega^2)
  \end{bmatrix}
  \begin{bmatrix}
  x_1 \\
  x_2
  \end{bmatrix}
  =
  \begin{bmatrix}
  D_e (-\omega^2) x_3 - \omega^2 D_o (-\omega^2) \\
  -D_o (-\omega^2) x_3 - D_e (-\omega^2)
  \end{bmatrix}.
  \]

- When $|A(\omega)| \neq 0$ for all $\omega > 0$,

  \[|A(\omega)| = \omega^2 N_o^2 (-\omega^2) + N_e^2 (-\omega^2) > 0, \quad \forall \omega > 0.\]
For every $x_3$, a unique solution $x_1$ and $x_2$ at each $\omega > 0$ given by:

$$
\begin{bmatrix}
  x_1 \\
  x_2 \\
\end{bmatrix} = \frac{1}{|A(\omega)|} \begin{bmatrix}
  N_o(-\omega^2) & N_e(-\omega^2) \\
  -N_e(-\omega^2) & \omega^2 N_o(-\omega^2) \\
\end{bmatrix} \begin{bmatrix}
  D_e(-\omega^2) x_3 - \omega^2 D_o(-\omega^2) \\
  -D_o(-\omega^2) x_3 - D_e(-\omega^2) \\
\end{bmatrix}
$$

or

$$
x_1(\omega) = \frac{1}{|A(\omega)|} \left( \left[ N_o(-\omega^2) D_e(-\omega^2) - N_e(-\omega^2) D_o(-\omega^2) \right] x_3 \\
-\omega^2 N_o(-\omega^2) D_o(-\omega^2) - N_e(-\omega^2) D_e(-\omega^2) \right)
$$

$$
x_2(\omega) = \frac{1}{|A(\omega)|} \left( \left[ -N_e(-\omega^2) D_e(-\omega^2) - \omega^2 N_o(-\omega^2) D_o(-\omega^2) \right] x_3 \\
+\omega^2 N_e(-\omega^2) D_o(-\omega^2) - \omega^2 N_o(-\omega^2) D_e(-\omega^2) \right).
$$

For a fixed value of $x_3$, let $\omega$ run from 0 to $\infty$. The above equations trace out a curve in the $x_1$ - $x_2$ plane corresponding to the complex root space boundary.
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- These curves along with the straight lines from the real root boundary conditions partition the parameter space into a set of open root distribution invariant regions. By sweeping over $x_3$, we find these regions.

- We now consider the case when $|A(\omega)| = 0$.

- Let

$$|A(\omega)| = \omega^2 N_o^2 (-\omega^2) + N_e^2 (-\omega^2) = 0, \text{ for some } \omega \neq 0. \ (\star)$$

- Since $N_o^2 (-\omega^2), N_e^2 (-\omega^2) \geq 0$, $(\star)$ holds if and only if

$$N_o (\omega^2) = N_e (-\omega^2) = 0.$$  

- Recall the matrix equation

$$\begin{bmatrix}
\omega^2 N_o (-\omega^2) & -N_e (-\omega^2) \\
N_e (-\omega^2) & N_o (-\omega^2)
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} = \begin{bmatrix}
D_e (-\omega^2) x_3 - \omega^2 D_o (-\omega^2) \\
-D_o (-\omega^2) x_3 - D_e (-\omega^2)
\end{bmatrix}. $$
• It means

\[ D_e (-\omega^2) x_3 - \omega^2 D_o (-\omega^2) = 0, \quad -D_o (-\omega^2) x_3 - D_e (-\omega^2) = 0 \]

• This is equivalent to

\[ \omega^2 D_o^2 (-\omega^2) + D_e (-\omega^2) = 0. \quad (*) \]

• Since \( D_o^2 (\omega^2) \), \( D_e^2 (-\omega^2) \geq 0 \), \((*)\) holds iff

\[ D_o (\omega^2) = D_e (-\omega^2) = 0. \]

• It follows that \(|A(\omega)| = 0\) has a solution for \( \omega \neq 0 \) iff

\[ N_o (\omega^2) = N_e (-\omega^2) = 0 \]
\[ D_o (\omega^2) = D_e (-\omega^2) = 0 \]

or \( D(s) \) and \( N(s) \) have a common factor \( s^2 + \omega^2 \) and this is ruled out by the assumption of stabilizability of the plant.
Example

- Consider the following 13th order plant

\[
P(s) = \frac{s^{10} + 2s^9 + 3s^8 + 4s^7 + 10s^6 + 5s^5 + s^4 - 7s^3 + 4s^2 + s + 23}{s^{13} + 9s^{12} + 40s^{11} + 111s^{10} + 203s^9 + 115s^8 - 203s^7 + 60s^6 + 25s^5 + s^4 - 18s^3 + 21s^2 + 2s + 7}
\]

- 1st order controller: \( C(s) = \frac{x_1 s + x_2}{s + x_3} \)

- The characteristic polynomial \( \delta(j\omega) = \delta_r(\omega) + j\omega\delta_i(\omega) \) where

\[
\delta_r(\omega) = -\omega^{14} + (40 + 9x_3)\omega^{12} + (-203 - 2x_1 - x_2 - 111x_3)\omega^{10} \\
+ (-203 + 4x_1 + 3x_2 + 115x_3)\omega^8 + (-25 - 5x_1 - 10x_2 - 60x_3)\omega^6 \\
+ (-18 - 7x_1 + x_2 + x_3)\omega^4 + (-2 - x_1 - 4x_2 - 21x_3)\omega^2 + (23x_2 + 7x_3)\\
\]

\[
\delta_i(\omega) = (9 + x_3)\omega^{12} + (-111 - x_1 - 40x_3)\omega^{10} + (115 + 3x_1 + 2x_2 + 203x_3)\omega^8 \\
+ (-60 - 10x_1 - 4x_2 + 203x_3)\omega^6 + (1 + x_1 + 5x_2 + 25x_3)\omega^4 \\
+ (-21 - 4x_1 + 7x_2 + 18x_3)\omega^2 + (7 + 23x_1 + x_2 + 2x_3) .
\]
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- For $\omega = 0$, $23x_2 + 7x_3 = 0$,
- For $\omega > 0$,

\[
x_1(\omega) = \frac{-9\omega^{22} + 120\omega^{20} - 181\omega^{18} - 452\omega^{16} + 1429\omega^{14} - 1738\omega^{12} + 3355\omega^{10} - 2931\omega^8 + 1586\omega^6 - 142\omega^4 + 504\omega^2 - 161}{|A(\omega)|} \\
+ \frac{9\omega^{22} - 120\omega^{20} + 280\omega^{18} - 805\omega^{16} + 406\omega^{14} - 1341\omega^{12} + 3501\omega^{10} - 3319\omega^8 + 1289\omega^6 - 225\omega^4 + 539\omega^2 - 154}{|A(\omega)|} \cdot x_3
\]

\[
x_2(\omega) = \frac{-9\omega^{24} + 120\omega^{22} - 280\omega^{20} + 805\omega^{18} - 406\omega^{16} + 1341\omega^{14} - 3501\omega^{12} + 3319\omega^{10} - 1289\omega^8 + 225\omega^6 - 539\omega^4 + 154\omega^2}{|A(\omega)|} \\
+ \frac{-27\omega^{22} + 396\omega^{20} - 1257\omega^{18} + 2596\omega^{16} - 1527\omega^{14} + 1042\omega^{12} - 2951\omega^{10} + 3303\omega^8 - 1364\omega^6 + 86\omega^4 - 518\omega^2 + 161}{|A(\omega)|} \cdot x_3
\]

where

\[
|A(\omega)| = \omega^{20} - 2\omega^{18} + 13\omega^{16} - 26\omega^{14} + 102\omega^{12} - 117\omega^{10} + 281\omega^8 \\
- 409\omega^6 + 76\omega^4 - 183\omega^2 + 529.
\]
For a fixed $x_3$
**EXAMPLE**

- **Plant:**
  
  \[ P(s) := \frac{3s^7 + 99s^6 + 1320s^5 + 9255s^4 + 37287s^3 + 88656s^2 + 120420s + 75600}{s^8 + 18s^7 + 131s^6 + 625s^5 + 2017s^4 + 4753s^3 + 7896s^2 + 8919s + 5670} \]

- **1st Order Controller:**
  
  \[ C(s) = \frac{x_1 s + x_2}{s + x_3} \]

- **Then we have**
  
  \[
  \begin{align*}
  N_0(s) &= 99s^6 + 9255s^4 + 88656s^2 + 75600 \\
  N_o(s) &= 3s^6 + 1320s^4 + 37287s^2 + 120420 \\
  D_e(s) &= s^8 + 131s^6 + 2017s^4 + 7896s^2 + 5670 \\
  D_o(s) &= 18s^6 + 625s^4 + 4753s^2 + 8919 \\
  \end{align*}
  \]

- **The characteristic polynomial**
  
  \[
  \delta(j\omega) = \delta_r(\omega) + j\omega\delta_i(\omega)
  \]

  \[
  \begin{align*}
  \delta_r(\omega) &= (18 + 3x_1 + x_3)\omega^8 + (-625 - 1320x_1 - 99x_2 - 131x_3)\omega^6 + (4753 + 37287x_1 + 9255x_2 + 2017x_3)\omega^4 \\
  &\quad + (-8919 - 120420x_1 - 88656x_2 - 7896x_3)\omega^2 + 75600x_2 + 5670x_3 \\
  \delta_i(\omega) &= \omega^8 + (-131 - 99x_1 - 3x_2 - 18x_3)\omega^6 + (2017 + 9255x_1 + 1320x_2 + 625x_3)\omega^4 \\
  &\quad + (-7896 - 88656x_1 - 37287x_2 - 4753x_3)\omega^2 + (5670 + 75600x_1 + 120420x_2 + 8919x_3) \\
  \end{align*}
  \]
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- For $\omega = 0$, $75600x_2 + 5670x_3 = 0$,

- For $\omega > 0$,

\[
x_1(\omega) = \frac{45\omega^{14} + 3411\omega^{12} - 9681\omega^{10} + 634077\omega^8 - 1899129\omega^6 - 69813\omega^4 + 25591140\omega^2 - 428652000}{|A(\omega)|} \\
\quad + \frac{-3\omega^{14} - 69\omega^{12} + 12207\omega^{10} - 159585\omega^8 + 220167\omega^6 - 6387621\omega^4 - 12203946\omega^2 + 8505000}{|A(\omega)|} \cdot x_3
\]

\[
x_2(\omega) = \frac{\omega^{16} + 69\omega^{14} - 12207\omega^{12} + 159585\omega^{10} - 220167\omega^8 + 6387621\omega^6 + 12203946\omega^4 - 8505000\omega^2}{|A(\omega)|} \\
\quad + \frac{-153\omega^{14} + 47859\omega^{12} - 3011169\omega^{10} + 62911227\omega^8 - 526622253\omega^6 + 1809907839\omega^4 - 2173643100\omega^2 + 428652000}{|A(\omega)|} \cdot x_3.
\]

where

\[
|A(\omega)| = 9\omega^{14} + 1881\omega^{12} + 133632\omega^{10} + 4048713\omega^8 + 52237809\omega^6 \\
\quad + 279041256\omega^4 + 1096189200\omega^2 + 5715360000.
\]
First Order Controllers for LTI Systems

Stability region for $x_3 = 0.2$

Stability region for $0.7 \leq x_3 \leq 0.5$
**ROBUST STABILIZATION BY FOCs**

- Robust stabilization problem considers the plant $P(s) = \frac{N(s)}{D(s)}$ where $P(s)$ is an interval plant.

- Consider a polynomial family of the form $\delta(s) = F_1(s)D(s) + F_2(s)N(s)$ where $N(s)$ and $D(s)$ are interval polynomials and

  $$F_i(s) = s^{t_i}(a_is + b_i)U_i(s)Q_i(s), \quad i = 1, 2.$$

- $t_i \geq 0$ is an arbitrary integer, $a_i$ and $b_i$ are arbitrary real, $U_i(s)$ is an anti-Hurwitz, and $Q_i(s)$ is an even or odd polynomial.

- Let $\Delta_v(s) := \{\delta_v(s) : F_1(s)D_i(s) + F_2(s)N_j(s), \quad i, j = 1, 2, 3, 4\}$ where $D_i(s)$ and $N_j(s)$ are the Kharitonov polynomials of $D(s)$ and $N(s)$, respectively.
Generalized Kharitonov Theorem

For robust stability of a family $\delta(s)$, it is enough that $F(s) = (F_1(s), F_2(s))$ stabilizes the finite set of vertex polynomials $\Delta_v(s)$.

- For the problem of robust stabilization with 1st order controllers, we consider
  
  $F_1(s) = s + x_3$ \quad and \quad $F_2(s) = x_1 s + x_2$.

- For the given interval plant
  
  $P(s) := \left\{ P(s) = \frac{N(s)}{D(s)} : N(s) \in N(s), \ D(s) \in D(s) \right\}$

- Let its vertices be
  
  $P_v(s) := \left\{ \frac{N_i(s)}{D_j(s)} : N_i(s) \in K_N(s), \ D_j(s) \in K_D(s) \right\}$

where $K_N(s)$ and $K_D(s)$ are the set of Kharitonov polynomials of $N(s)$ and $D(s)$, respectively.
Robust Stabilization with FOC

Let $R_k$ be controller parameter space regions that consist of all first order stabilizing controllers for the $k^{th}$ vertex system $P_k(s)$. Then every first order stabilizing controller that robustly stabilizes the interval system $P(s)$ is given by

$$\mathcal{R} := \cap_k R_k.$$
Example

- Plant $P(s) = \left\{ \frac{N(s)}{D(s)} \right\} = \left\{ \frac{n_7 s^7 + n_6 s^6 + n_5 s^5 + n_4 s^4 + n_3 s^3 + n_2 s^2 + n_1 s + n_0}{d_8 s^8 + d_7 s^7 + d_6 s^6 + d_5 s^5 + d_4 s^4 + d_3 s^3 + d_2 s^2 + d_1 s + d_0} \right\}$

- The upper and lower bounds of the coefficients of interval polynomials $D(s)$ and $N(s)$ are

$$
D^+ = \{2, 18.1, 131.1, 625.1, 2017.1, 4753.1, 7896.1, 8919.1, 5670.1\} \\
D^- = \{1, 17.9, 130.9, 624.9, 2016.9, 4752.9, 7895.9, 8918.9, 5669.9\} \\
N^+ = \{3.1, 99.1, 1320.1, 9255.1, 37287.1, 88656.1, 120420.1, 75600.1\} \\
N^- = \{2.9, 98.9, 1319.9, 9254.9, 37286.9, 88655.9, 120419.9, 75599.9\}
$$

- We now construct the 16 vertex systems:

$$
P_v(s) = \left\{ P_l(s) : \frac{N_i(s)}{D_j(s)}, \ N_i(s) \in \mathcal{K}_N(s), \ D_j(s) \in \mathcal{K}_D(s) \right\}
$$
First Order Controllers for LTI Systems

Stabilizing regions for all 16 vertex systems
• To observe the intersection of these 16 regions, we consider

Region for robust stability when \( x_3 = 0.5 \)

Area magnified
**$H_\infty$ DESIGN WITH FOC**

- The gain margin and phase margin specifications can be imposed by replacing the plant by $KP(s)$ and $e^{-j\theta}P(s)$ respectively.

- Consider the standard feedback configuration.

- The plant is assumed to be SISO, LTI, proper, and coprime.

- For a given closed loop transfer function $T(s)$ and positive scalar $\gamma$, define the $H_\infty$ design criteria to be:

$$\|W(s)T(s)\|_\infty < \gamma \quad \text{where} \quad W(s) = \frac{W_n(s)}{W_d(s)}$$

is a stable, coprime, frequency-dependent weighting function.
First Order Controllers for LTI Systems

- Consider the complementary sensitivity function

\[ T(s) = \frac{(x_1 s + x_2) N(s)}{(s + x_3) D(s) + (x_1 s + x_2) N(s)} . \]

- The objective is to determine the region in the parameter space for which the closed loop system is stable and the above defined \( H_\infty \) optimization criteria is satisfied.

- Specifically, the objective is to determine values of \( x_1, x_2, \) and \( x_3 \) (if any) for the controller \( C(s) \), such that

\[ \left\| \frac{W_n(s)}{W_d(s)} \frac{(x_1 s + x_2) N(s)}{(s + x_3) D(s) + (x_1 s + x_2) N(s)} \right\|_\infty < \gamma \]

and the closed loop characteristic polynomial

\[ \delta(s) := (s + x_3) D(s) + (x_1 s + x_2) N(s) \in \mathcal{H} \]
LEMMA

Let \( F(s) = \frac{N_F(s)}{D_F(s)} \) be a stable and proper rational function, where \( N_F(s) \) and \( D_F(s) \) are polynomials with \( \text{deg}[D_F(s)] = q \) and \( n_q, d_q \) denote the \( q^{th} \) degree coefficients of \( N_F, D_F \). Define \( \phi(s) := D_F(s) + \frac{1}{\gamma} e^{j\theta} N_F(s) \).

Then for a given \( \gamma > 0 \), \( \|F(s)\|_\infty < \gamma \) if and only if

1. \( |n_q| < \gamma |d_q| \)

2. \( \phi(s) \) is Hurwitz for all \( \theta \) in \([0, 2\pi)\),

where \( n_q \) and \( d_q \) are the coefficients of \( s^q \) in \( N_F(s) \) and \( D_F(s) \), respectively.
• We now apply this result to our problem.

• Set

\[ N_F(s) = W_n(s) (x_1 s + x_2) N(s) \]
\[ D_F(s) = W_d(s) \left[ (s + x_3) D(s) + (x_1 s + x_2) N(s) \right] . \]

• Let

\[ W_n(s) = a_m s^m + a_{m-1} s^{m-1} + \cdots + a_1 s + a_0 \]
\[ W_d(s) = b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0 \]
\[ N(s) = \alpha_n s^n + \alpha_{n-1} s^{n-1} + \cdots + \alpha_1 s + \alpha_0 \]
\[ D(s) = \beta_n s^n + \beta_{n-1} s^{n-1} + \cdots + \beta_1 s + \beta_0 \]

where \( b_m \) and \( \beta_n \neq 0 \)

• Then \( n_q = \alpha_n a_m x_1 \) and \( d_q = b_m (\beta_n + \alpha_n x_1) \).
Problem Formulation

Determine the region in the controller parameter space such that

\[ \delta(s) := (s + x_3)D(s) + (x_1s + x_2)N(s) \in \mathcal{H} \]

\[ |n_q| < \gamma|d_q| \text{ and} \]

\[ \phi(s) := D_F(s) + \frac{1}{\gamma}e^{j\theta}N_F(s) \in \mathcal{H} \text{ for all } \theta \in [0, 2\pi). \]

The problem of determining the controller to satisfy the above simultaneous stabilization conditions can be solved as before by the D-decomposition technique with fixed \( x_3 \).
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Example

- Plant and controller: \( G(s) = \frac{s - 1}{s^2 + 0.8s - 0.2} \) and \( C(s) = \frac{x_1 s + x_2}{s + x_3} \)

- The closed loop transfer function is

\[
T(s) = \frac{(x_1 s + x_2)(s - 1)}{(s + x_3)(s^2 + 0.8s - 0.2) + (x_1 s + x_2)(s - 1)}.
\]

- We arbitrarily choose \( x_3 = 2.5, \gamma = 1 \), and the weighting function to be the high pass transfer function

\[
W(s) = \frac{s + 0.1}{s + 1}.
\]

- Determine the set of all stabilizing controllers, \( C(s) \) such that \( \|W(s)T(s)\|_\infty < 1 \) for \( x_3 = 2.5 \).
First, determine the stability region

Note that the plant is strictly proper so that $|n_\alpha| < \gamma |d_\alpha|$ for all $x_1$, $x_2$, and $x_3$.

Determine the region for which $\phi(s)$ is Hurwitz for all $\theta \in [0, 2\pi)$ where

$$
\phi(s) = (s + 1) \left[ (s + x_3) \left( s^2 + 0.8s - 0.2 \right) + (x_1 s + x_2) (s - 1) \right] 
+ e^{j\theta} (s + 0.1) (x_1 s + x_2) (s - 1).
$$

Using the substitution, $e^{j\theta} = \alpha + j\beta$,

$$
\phi(j\omega) = \omega^4 + \beta x_1 \omega^3 + (-0.6 + 0.9\alpha x_1 - x_2 - \alpha x_2 - 1.8x_3) + (0.1\beta x_1 + 0.9\beta x_2) \omega \\
- (1 + 0.1\alpha)x_2 - 0.2x_3 \\
+ j \left[ (-1.8 - x_1 - \alpha x_1 - x_3) \omega^3 + (0.9\beta x_1 - \beta x_2) \omega^2 \\
+ (-0.2 - x_1 - 0.1\alpha x_1 - 0.9\alpha x_2 + 0.6x_3) \omega - 0.1\beta x_2 \right].
$$
First, we consider the real root boundaries. The real root boundary for the origin is obtained by setting \( \phi(0) = 0 \):

\[
-(1 + 0.1\alpha)x_2 - 0.2x_3 = 0, \quad -0.1\beta x_2 = 0
\]

Since the plant is strictly proper, there is not a real root boundary at infinity.

The complex root boundary is characterized by \( \phi(j\omega) = 0 \).
First Order Controllers for LTI Systems

stability region satisfying $H_\infty$ constraint

stability region
FO DISCRETE-TIME CONTROLLERS

- Plant and controllers:
  \[ P(z) = \frac{N(z)}{D(z)} \]
  \[ C(z) = \frac{x_1 z + x_2}{z + x_3} = \frac{N_c(z)}{D_c(z)} \]

- The characteristic polynomial is
  \[ \delta(z) = D_c(z)D(z) + N_c(z)N(z) \]
  \[ = (z + x_3)D(z) + (x_1 z + x_2) N(z) \]
First Order Discrete-time Controllers

- We now determine the image of a polynomial
  \[ P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_2 z^2 + a_1 z + a_0 \]
  with real coefficients, evaluated on the unit circle.

- Setting
  \[ u := -\cos \theta, \]
  we have \[ z^k = e^{jk\theta} = \cos k\theta + j \sin k\theta. \]

- Using Tchebyshev representations, we have
  \[ P(e^{j\theta}) = R_P(u) + j \sqrt{1-u^2} T_P(u) \]
  where \[ R_P(u) = a_n c_n(u) + a_{n-1} c_{n-1}(u) + \cdots + a_1 c_1(u) + a_0 \]
  \[ T_P(u) = a_n s_n(u) + a_{n-1} s_{n-1}(u) + \cdots + a_1 s_1(u). \]

are real polynomials of degree \( n \) and \( n-1 \) respectively.
First Order Discrete-time Controllers

- Consider the characteristic polynomial
  \[ \delta(z) = (z + x_3) D(z) + (x_1 z + x_2) N(z) \]
- Then we write
  \[
  D(e^{j\theta}) := R_D(u) + j\sqrt{1-u^2} T_D(u), \quad N(e^{j\theta}) := R_N(u) + j\sqrt{1-u^2} T_N(u)
  \]
  \[
  e^{j\theta} + x_3 = -u + j\sqrt{1-u^2} + x_3 \\
  x_1 e^{j\theta} + x_2 = -x_1 u + j\sqrt{1-u^2} x_1 + x_2.
  \]
- The characteristic polynomial evaluated on the unit circle then becomes
  \[ \Pi(u) = \Pi_r(u) + j\Pi_i(u) \]
  where
  \[
  \Pi_r(u) = R_D(u) (x_3 - u) - T_D(u) (1 - u^2) + R_N(u) (x_2 - ux_1) - (1-u^2) x_1 T_N(u)
  \]
  \[
  \Pi_i(u) = \sqrt{1-u^2} [R_D(u) + T_D(u) (x_3 - u) + x_1 R_N(u) + T_N(u) (x_2 - ux_1)]
  \]
First Order Discrete-time Controllers

- The stability boundary for complex roots is given by setting
  \[ \Pi(u) = 0, \quad u \in (-1, 1) \]

- The stability boundaries for real roots are given by
  \[ \Pi(-1) = 0, \quad \Pi(1) = 0. \]

- The complex root boundary from the Boundary Crossing Conditions is given by
  \[ \Pi_r(u) = 0, \quad \Pi_i(u) = 0 \quad (**) \]

- Note that at \( u = 1 \) and \( u = -1 \), \( (**) \) is trivially satisfied, that is holds for all \( x_1, x_2, \) and \( x_3 \).

- At \( u = 1 \), \( (*) \) becomes
  \[ R_D(1)(x_3 - 1) + R_N(1)(x_2 - x_1) = 0 \]
  which for a given \( x_3 \) is a straight line in the \( x_1 - x_2 \) plane.
First Order Discrete-time Controllers

- Similarly, at \( u = -1 \),
  \[
  R_D(-1) (x_3 + 1) + R_N(-1) (x_2 + x_1) = 0
  \]
  which for fixed \( x_3 \) is a straight line.

- For \(-1 < u < 1\),
  \[
  \Pi_r(u) = 0, \quad \Pi_i(u) = 0
  \]

\[
\begin{bmatrix}
(u^2 - 1) T_N(u) - u R_N(u) & R_N(u) \\
R_N(u) - u T_N(u) & T_N(u)
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
= \begin{bmatrix}
(u - x_3) R_D(u) + (1 - u^2) T_D(u) \\
(u - x_3) T_D(u) - R_D(u)
\end{bmatrix}
\]

and
\[
|A(u)| = T_N^2(u) (u^2 - 1) - R_N^2(u).
\]
First Order Discrete-time Controllers

- A unique solution $x_1$ and $x_2$ at each $u \in (-1, 1)$ is obtained from

\[
\begin{align*}
x_1(u) &= \frac{Y(u)(u - x_3) + (1 - u^2)T_D(u)T_N(u) + R_D(u)R_N(u)}{|A(u)|} \\
x_2(u) &= \frac{Y(u)(1 - ux_3) + x_3[R_N(u)R_D(u) + (1 - u^2)T_N(u)T_D(u)]}{|A(u)|}
\end{align*}
\]

where $Y(u) = R_D(u)T_N(u) - R_N(u)T_D(u)$.

- The above two equations trace out a curve in the $x_1 - x_2$ plane representing the complex root space boundary, for fixed $x_3$, as $u$ runs from -1 to +1.

- This curve along with the lines and partition the controller parameter space into regions with a fixed number of outside the unit circle roots.

- By sweeping over $x_3$, we can identify the three dimensional stability region for a given plant, if one exists.
Example

- Consider the following 6th order discrete-time plant

\[
P(z) = \frac{24z^5 + 72z^4 + 19z^3 + 81z^2 + 84z + 95}{76z^6 + 42z^5 + 56z^4 + 59z^3 + 24z^2 + z + 15}
\]

- A controller

\[
C(z) = \frac{x_1 z + x_2}{z + x_3}
\]

- Choosing \(x_3 = 0.75\), the lines corresponding to \(u = -1\) and \(u = 1\) are

\[u = -1:\]
\[x_2 = -x_1 + (1 + x_3) \left[ \frac{273}{375} \right] = -x_1 - 1.274\]

\[u = 1:\]
\[x_2 = x_1 + (1 - x_3) \left[ \frac{69}{121} \right] = x_1 + 0.14256.\]
• For $-1 < u < 1$, we have the curve given parametrically by

\[
\begin{align*}
x_1(u) &= \frac{462080u^6 - 505248u^5 - 217368u^4 + 303488u^3 + 13524u^2 - 38088u - 3014.25}{|A(u)|} \\
x_2(u) &= \frac{222400u^5 - 233616u^4 - 54554u^3 + 86446u^2 - 3246u - 4143.5}{|A(u)|}
\end{align*}
\]

where

\[|A(u)| = 72960u^5 - 141696u^4 - 12824u^3 + 79380u^2 + 2856u - 15317.\]

• Figure next illustrates how the curve and lines partition the controller parameter space for a fixed $x_3 = 0.75$. Despite the simple controller structure, the behavior of the curve is extremely complicated.
First Order Discrete-time Controllers

stability region for $-1.25 \leq x_3 \leq 1.375$

stability region with $x_3 = 0.75$

- The figure shows that the region gets smaller as $x_3$ approaches $+1.375$ and $-1.25$. 
Extension to Design 1\textsuperscript{st} Order Controllers Achieving Maximum Delay Tolerance

- Delay tolerance can be accomplished by solving the problem of simultaneously stabilizing the systems

\[
P(z), \quad z^{-1}P(z), \quad z^{-2}P(z), \quad \cdots, \quad z^{-q}P(z).
\]

Example

- Consider the plant given in the previous example with \(x_3 = -0.25\).

- As shown, the stability region shrinks as \(q\) increases.
First Order Discrete-time Controllers

- **no delay**
- **with one delay**
- **with one and two delays**
- **with one, two, and three delays**
Maximally Deadbeat Design

- This problem can be solved by obtaining the Tchebychev representation of the characteristic equation on a circle of radius $\rho$.
- By reducing the value of $\rho$ from 1 we can determine the value of $\rho = \rho^*$ for which the set of stabilizing first order controllers just becomes empty.

Example

- Consider the plant - controller pair of the previous example.

- The “stabilizing region” for $x_3 = -0.25$ and various values of $\rho$ are shown in the figure. The stabilizing region vanishes for $\rho = 0.8$.  

First Order Controllers for LTI Systems

A point selected is (0.3249,-0.4737) Controller $C_1(z)$

A point selected is (0.0518,-0.1184) Controller $C_2(z)$

- Step responses illustrate the approximate deadbeat property

with $C_2(z)$

with $C_1(z)$