Digital PID Controller Design
Digital PID Controller Design

- Plant and Controller

\[ G(z) = \frac{N(z)}{D(z)} , \quad C(z) = \frac{N_C(z)}{D_C(z)} . \]

- The characteristic polynomial of the closed loop system

\[ \Pi(z) := D_C(z)D(z) + N_C(z)N(z) \]
TCHEBYSHEV REPRESENTATION AND ROOT CLUSTERING

Tchebyshev representation of real polynomials

- Consider a real polynomial $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$

- The image of $P(z)$ evaluated on the circle $C_\rho$ of radius $\rho$, centered at the origin is:

\[
\{ P(z) : z = \rho e^{j\theta}, \quad 0 \leq \theta \leq 2\pi \}.
\]

- As the coefficients $a_i$ are real $P(\rho e^{j\theta})$ and $P(\rho e^{-j\theta})$ are conjugate complex numbers, and so it suffices to determine the image of the upper half of the circle:

\[
\{ P(z) : z = \rho e^{j\theta}, \quad 0 \leq \theta \leq \pi \}.
\]
• Since \( z^k \big|_{z=\rho e^{i\theta}} = \rho^k (\cos k\theta + j \sin k\theta) \),

\[
P (\rho e^{j\theta}) = \left( a_n \rho^n \cos n\theta + \cdots + a_1 \rho \cos \theta + a_0 \right) + j \left( a_n \rho^n \sin n\theta + \cdots + a_1 \rho \sin \theta \right)
\]

\[
= \bar{R}(\rho, \theta) + j \bar{I}(\rho, \theta).
\]

• Consider \((\rho e^{j\theta})^k = \rho^k \cos k\theta + j \rho^k \sin k\theta\)

• Write \( u = -\cos \theta \) and define the generalized Tchebyshev polynomials as follows:

\[
c_k(u, \rho) = \rho^k c_k(u), \quad s_k(u, \rho) = \rho^k s_k(u), \quad k = 0, 1, 2 \cdots
\]

and note that

\[
s_k(u, \rho) = -\frac{1}{k} \cdot \frac{d}{du} [c_k(u, \rho)], \quad k = 1, 2, \cdots
\]

\[
c_{k+1}(u, \rho) = -\rho u c_k(u, \rho) - \left(1 - u^2\right) \rho s_k(u, \rho), \quad k = 1, 2, \cdots
\]
The generalized Tchebyshev polynomials are for \( k = 1, \ldots, 5 \):

<table>
<thead>
<tr>
<th>( k )</th>
<th>( c_k(u, \rho) )</th>
<th>( s_k(u, \rho) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(-\rho u)</td>
<td>( \rho )</td>
</tr>
<tr>
<td>2</td>
<td>( \rho^2 (2u^2 - 1) )</td>
<td>(-2\rho^2 u)</td>
</tr>
<tr>
<td>3</td>
<td>( \rho^3 (-4u^3 + 3u) )</td>
<td>( \rho^3 (4u^2 - 1) )</td>
</tr>
<tr>
<td>4</td>
<td>( \rho^4 (8u^4 - 8u^2 + 1) )</td>
<td>( \rho^4 (-8u^3 + 4u) )</td>
</tr>
<tr>
<td>5</td>
<td>( \rho^5 (-16u^5 + 20u^3 - 5u) )</td>
<td>( \rho^5 (16u^4 - 12u^2 + 1) )</td>
</tr>
<tr>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
</tr>
</tbody>
</table>

With this notation, \( P(\rho e^{j\theta}) = R(u, \rho) + j\sqrt{1-u^2}T(u, \rho) =: P_c(u, \rho) \) where

\[
R(u, \rho) = a_n c_n(u, \rho) + a_{n-1} c_{n-1}(u, \rho) + \cdots + a_1 c_1(u, \rho) + a_0
\]
\[
T(u, \rho) = a_n s_n(u, \rho) + a_{n-1} s_{n-1}(u, \rho) + \cdots + a_1 s_1(u, \rho).
\]

- \( R(u, \rho) \) and \( T(u, \rho) \) are polynomials in \( u \) and \( \rho \).

- The complex plane image of \( P(z) \) as \( z \) traverses the upper half of the circle \( \mathcal{C}_\rho \) can be obtained by evaluating \( P_c(u, \rho) \) as \( u \) runs from \(-1 \) to \(+1\).
**LEMMA**

For a fixed \( \rho > 0 \),

(a) if \( P(z) \) has no roots on the circle of radius \( \rho > 0 \),
   \( (R(u, \rho), T(u, \rho)) \) have no common roots for \( u \in [-1, 1] \) and \( R(\pm 1, \rho) \neq 0 \).

(b) if \( P(z) \) has \( 2m \) roots at \( z = -\rho (z = +\rho) \),
   then \( R(u, \rho) \) and \( T(u, \rho) \) have \( m \) roots each at \( u = +1 \) \( (u = -1) \).

(c) if \( P(z) \) has \( 2m - 1 \) roots at \( z = -\rho (z = +\rho) \), then \( R(u, \rho) \) and \( T(u, \rho) \)
   have \( m \) and \( m - 1 \) roots, respectively at \( u = +1 \) \( (u = -1) \).

(d) if \( P(z) \) has \( q_i \) pairs of complex roots at \( z = -\rho u_i \pm j\rho \sqrt{1 - u_i^2} \), for \( u_i \neq \pm 1 \),
   then \( R(u, \rho) \) and \( T(u, \rho) \) each have \( q_i \) real roots at \( u = u_i \).

- When the circle of interest is the unit circle, that is \( \rho = 1 \),
  we will write \( P_c(u, 1) = P_c(u) \) and also

\[
R(u, 1) =: R(u), \quad T(u, 1) =: T(u).
\]
Interlacing Conditions for Root Clustering and Schur Stability

THEOREM

$P(z)$ has all its zeros strictly within $C_\rho$ if and only if

(a) $R(u, \rho)$ has $n$ real distinct zeros $r_i$, $i = 1, 2, \cdots, n$ in $(-1, 1)$.

(b) $T(u, \rho)$ has $n - 1$ real distinct zeros $t_j$, $j = 1, 2, \cdots, n - 1$ in $(-1, 1)$.

(c) The zeros $r_i$ and $t_j$ interlace:

$$-1 < r_1 < t_1 < r_2 < t_2 < \cdots < t_{n-1} < r_n < +1.$$ 

The three conditions given in the above Theorem may be referred to as interlacing conditions on $R(u, \rho)$ and $T(u, \rho)$. By setting $\rho = 1$ in the above Theorem we obtain conditions for Schur stability in terms of interlacing of the zeros of $R(u)$ and $T(u)$. 
Tchebyshev Representation of Rational Functions

Let

\[ P_i(z) \big|_{z=-\rho u+j\rho \sqrt{1-u^2}} = R_i(u, \rho) + j\sqrt{1-u^2}T_i(u, \rho), \quad i=1,2. \]

\[ Q(z) \big|_{z=-\rho u+j\rho \sqrt{1-u^2}} = \frac{P_1(z)P_2(z^{-1})}{P_2(z)P_2(z^{-1})} \bigg|_{z=-\rho u+j\rho \sqrt{1-u^2}} \]

\[ = \frac{R(u, \rho)}{(R_1(u, \rho)R_2(u, \rho) + (1-u^2)T_1(u, \rho)T_2(u, \rho))} \]

\[ + j\sqrt{1-u^2} \frac{(T_1(u, \rho)R_2(u, \rho) - R_1(u, \rho)T_2(u, \rho))}{R_2^2(u, \rho) + (1-u^2)T_2^2(u, \rho)}. \]

\[ R(u, \rho), \quad T(u, \rho) \]

are rational functions of the real variable \( u \) which runs from -1 to +1.
ROOT COUNTING FORMULAS

LEMMA
Let the real polynomial $P(z)$ have $i$ roots in the interior of the circle $C_{\rho}$ and no roots on the circle. Then:

$$\Delta_0^\pi [\phi_P(\theta)] = \pi i$$

LEMMA
Let $Q(z) = \frac{P_1(z)}{P_2(z)}$, where the real polynomials $P_1(z)$ and $P_2(z)$ have $i_1$ and $i_2$ roots, respectively in the interior of the circle $C_{\rho}$ and no roots on the circle. Then

$$\Delta_0^\pi [\phi_Q(\theta)] = \pi (i_1 - i_2) = \Delta_{-1}^+ [\phi_{Q_C}(u)].$$
• Let \( t_1, \ldots, t_k \) denote the real distinct zeros of \( T(u, \rho) \) of odd multiplicity, for \( u \in (-1, 1) \), ordered as follows:
\(-1 < t_1 < t_2 < \cdots < t_k < +1\). Suppose also that \( T(u, \rho) \) has \( p \) zeros at \( u = -1 \) and let \( f^i(x_0) \) denote the \( i \)-th derivative to \( f(x) \) evaluated at \( x = x_0 \).

**THEOREM**

Let \( P(z) \) be a real polynomial with no roots on the circle \( C_\rho \) and suppose that \( T(u, \rho) \) has \( p \) zeros at \( u = -1 \). Then the number of roots \( i \) of \( P(z) \) in the interior of the circle \( C_\rho \) is given by

\[
i = \frac{1}{2} Sgn \left[ T^{(p)}(-1, \rho) \right] \left( Sgn \left[ R(-1, \rho) \right] + 2 \sum_{j=1}^{k} (-1)^j Sgn \left[ R(t_j, \rho) \right] + (-1)^{k+1} Sgn \left[ R(+1, \rho) \right] \right).
\]
• The result derived above can now be extended to the case of rational functions. Let $Q(z) = \frac{P_1(z)}{P_2(z)}$ where $P_i(z), i = 1, 2$ are real rational functions.

• Tchebyshev representation of $Q(z)$ on the circle $C_\rho$.

Let $R(u, \rho), T(u, \rho)$ be defined by:

\[
R(u, \rho) = R_1(u, \rho)R_2(u, \rho) + (1 - u^2)T_1(u, \rho)T_2(u, \rho)
\]
\[
T(u, \rho) = T_1(u, \rho)R_2(u, \rho) - R_1(u, \rho)T_2(u, \rho)
\]

• Suppose that $T(u, \rho)$ has p zeros at $u = -1$ and let $t_1 \cdots t_k$ denote the real distinct zeros of $T(u, \rho)$ of odd multiplicity ordered as $-1 < t_1 < t_2 < \cdots < t_k < +1$.

**THEOREM**

Let $Q(z) = \frac{P_1(z)}{P_2(z)}$ where $P_i(z), i = 1, 2$ are real polynomials with $i_1$ and $i_2$ zeros respectively inside the circle $C_\rho$ and no zeros on it. Then

\[
i_1 - i_2 = \frac{1}{2} \text{Sgn} [T^{(p)}(-1, \rho)] \left( \text{Sgn} [R(-1, \rho)] + 2 \sum_{j=1}^{k} (-1)^j \text{Sgn} [R(t_j, \rho)] + (1)^k \text{Sgn} [R(+1, \rho)] \right).
\]
DIGITAL PI, PD AND PID CONTROLLERS

- For PI controllers,

\[
C(z) = K_P + K_I T \cdot \frac{z}{z-1} = \frac{(K_P + K_I T) \left( z - \frac{K_P}{K_I T + K_P} \right)}{z - 1} = \frac{K_1 (z - K_2)}{z - 1}
\]

where \( K_P = K_1 K_2 \), \( K_I = \frac{K_1 - K_1 K_2}{T} \).

- For PD controllers,

\[
C(z) = K_P + \frac{K_D}{T} \cdot \frac{z - 1}{z} = \frac{(K_P + \frac{K_D}{T}) \left( z - \frac{K_D}{K_P + \frac{K_D}{T}} \right)}{z}
\]

\[= \frac{K_1 (z - K_2)}{z}\]

where \( K_P = K_1 - K_1 K_2 \), \( K_D = K_1 K_2 T \).

- The general formula of a discrete PID controller, using backward differences to preserve causality,

\[
C(z) = K_P + K_I T \cdot \frac{z}{z-1} + \frac{K_D}{T} \cdot \frac{z - 1}{z} = \frac{K_2 z^2 + K_1 z + K_0}{z(z - 1)}
\]

where

\[
K_P = -K_1 - 2K_0, \quad K_I = \frac{K_0 + K_1 + K_2}{T}, \quad K_D = K_0 T.
\]
COMPUTATION OF THE STABILIZING SET

Constant Gain Stabilization

- Plant $G(z) = \frac{N(z)}{D(z)}$

- The closed-loop characteristic polynomial is

$$\delta(z) = D(z) + KN(z).$$

- Tchebyshev representations of $D(z)$ and $N(z)$

$$D(e^{j\theta}) = R_D(u) + j\sqrt{1 - u^2}T_D(u)$$
$$N(e^{j\theta}) = R_N(u) + j\sqrt{1 - u^2}T_N(u),$$

- Note also that $N(e^{-j\theta}) = R_D(u) - j\sqrt{1 - u^2}T_D(u)$ and $N(z^{-1}) = \frac{N_r(z)}{z^l}$

where $N_r(z)$ is the reverse polynomial and $l$ is the degree of $N(z)$. 
\[ \delta(z)N(z^{-1}) = D(z)N(z^{-1}) + KN(z)N(z^{-1}) \]

\[
\frac{\delta(z)N_r(z)}{z^l} \bigg|_{z=e^{j\theta}} = \left( R_D(u) + j\sqrt{1-u^2}T_D(u) \right) \left( R_N(u) - j\sqrt{1-u^2}T_N(u) \right) \\
+ K \left[ R_N^2(u) + (1-u^2)T_N^2(u) \right] \\
= R_D(u)R_N(u) + (1-u^2)T_D(u)T_N(u) + K \left[ R_N^2(u) + (1-u^2)T_N^2(u) \right] \\
\]

\[
R(K,u) + j\sqrt{1-u^2} \left[ T_D(u)R_N(u) - R_D(u)T_N(u) \right] \\
T(u) \\
= R(K,u) + j\sqrt{1-u^2}T(u). \\
\]

- The imaginary part of the above expression has been rendered independent of \( K \) as a result of multiplying \( \delta(z) \) by \( N(z^{-1}) \).

**Digital PID Controller Design**

**Ready to apply the root counting formulas**
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Constant Gain Stabilization Algorithm

- Let $t_i, i = 1, 2, \cdots, k$ denote the real zeros of odd multiplicity of the fixed $T(u)$, for $u$ in $(-1, +1)$ and set $t_0 = -1, t_{k+1} = +1$.
- Write $\text{Sgn} [R(K, t_j)] = x_j, \quad j = 0, 1, \cdots, k + 1$
- Let $i_\delta, i_{N_r}$ denote the number of zeros of $\delta(z)$ and $N_r(z)$ inside the unit circle. For simplicity assume that $N(z)$ has no unit circle zeros and therefore neither does $N_r(z)$.

\[
i_\delta + i_{N_r} - l = \frac{1}{2} \text{Sgn} [T^{(p)}(-1)] \\
\cdot \left( \text{Sgn} [R(K, -1)] + 2 \sum_{j=1}^{k} (-1)^j \text{Sgn} [R(K, t_j)] + (-1)^{k+1} \text{Sgn} [R(K, +1)] \right).
\]
Example

\[ G(z) = \frac{z^4 + 1.93z^3 + 2.2692z^2 + 0.1443z - 0.7047}{z^5 - 0.2z^4 - 3.005z^3 - 3.9608z^2 - 0.0985z + 1.2311}. \]

- Then
  \[ R_D(u) = -16u^5 - 1.6u^4 + 32.02u^3 - 6.3216u^2 - 13.9165u + 4.9919 \]
  \[ T_D(u) = 16u^4 + 1.6u^3 - 24.02u^2 + 7.1216u + 3.9065 \]
  \[ R_N(u) = 8u^4 - 7.72u^3 - 3.4616u^2 + 5.6457u - 1.9739 \]
  \[ T_N(u) = -8u^3 + 7.72u^2 - 0.5384u - 1.7857 \]

- and
  \[ T(u) = T_D(u)R_N(u) - R_D(u)T_N(u) \]
  \[ = -11.2752u^4 + 7.5669u^3 + 16.7782u^2 - 14.1655u + 1.203. \]

- The roots of \( T(u) \) of odd multiplicity and lying in \((-1, 1)\) are 0.0963 and 0.8358.
Digital PID Controller Design

\[ R(K, u) = 11.2752u^5 + 12.1307u^4 - 40.6359u^3 - 7.1779u^2 + 40.8322u \\
\quad - 16.8293 - 19.6615u - 5.4727 \\
\quad + K \left( -11.2752u^4 + 9.7262u^3 + 15.0696u^2 - 20.3653u + 7.085 \right) . \]

- Since \( i_\delta = 5 \) for stability, and \( i_{N_r} = 2 \) and \( l = 4 \), we must have:

\[
\text{Sgn} \left[ T^{(p)}(-1) \right] \left( \text{Sgn}[R(K, -1)] - 2\text{Sgn}[R(K, 0.0963)] + 2\text{Sgn}[R(K, 0.8358)] - \text{Sgn}[R(K, 1)] \right) = 6
\]

- Since \( \text{Sgn} \left[ T^{(p)}(-1) \right] = +1 \), we have the only feasible string given by:

\[
\begin{array}{ccccc}
\text{Sgn}[R(K,-1)] & \text{Sgn}[R(K, 0.0963)] & \text{Sgn}[R(K, 0.8358)] & \text{Sgn}[R(K, 1)] \\
1 & -1 & 1 & -1
\end{array}
\]
This translates into the following set of inequalities:

\[ R(K, -1) = -23.348 + 21.5185K > 0 \Rightarrow K > 1.085 \]
\[ R(K, 0.0963) = -12.998 + 5.2709K < 0 \Rightarrow K < 2.466 \]
\[ R(K, 0.8358) = -0.9232 + 0.7673K > 0 \Rightarrow K > 1.2032 \]
\[ R(K, 1) = -0.4050 + 0.2403K < 0 \Rightarrow K < 1.6854. \]

The closed loop system is stable for \( 1.2032 < K < 1.6854 \).

In this example, we have \( x_j, j = 0, 1, 2, 3 \). Each \( x_j \) may assume the value +1 or −1 since 0 is excluded as we are testing for stability. This leads to \( 2^4 = 16 \) possible strings which may satisfy the signature requirement. In this example, only one string of the possible 16 satisfies the signature requirement.
Stabilization with PI Controllers

- Plant and Controller: \( P(z) = \frac{N(z)}{D(z)}, \quad C(z) = \frac{K_1(z - K_2)}{z - 1} \)

- The characteristic polynomial: \( \delta(z) = (z - 1)D(z) + K_1(z - K_2)N(z) \)

- Writing the Tchebyshev representations of \( D(z), N(z) \) and \( N(z^{-1}) \)

- Then to achieve parameter separation, we calculate

\[
\delta(z)N(z^{-1}) \mid_{u = -\cos \theta} = \left(-u - 1 + j\sqrt{1-u^2}\right)\left(P_1(u) + j\sqrt{1-u^2}P_2(u)\right) + jK_1\sqrt{1-u^2}P_3(u) - K_1(u + K_2)P_3(u)
\]

where

\[
P_1(u) = R_D(u)R_N(u) + (1 - u^2)T_D(u)T_N(u)
\]

\[
P_2(u) = R_N(u)T_D(u) - T_N(u)R_D(u)
\]

\[
P_3(u) = R_N^2(u) + (1 - u^2)T_N^2(u).
\]
\[
\delta(z) N(z^{-1}) \bigg|_{z=e^{j\theta}, u=-\cos \theta} = \frac{\delta(z) N_r(z)}{z^l} \bigg|_{z=e^{j\theta}, u=-\cos \theta} \\
= R(u, K_1, K_2) + \sqrt{1 - u^2} T(u, K_1)
\]

where
\[
R(u, K_1, K_2) = -(u+1)P_1(u) - (1-u^2)P_2(u) - K_1(u+K_2)P_3(u)
\]
\[
T(u, K_1) = P_1(u) - (u+1)P_2(u) + K_1P_3(u).
\]

- For a fixed value of $K_1$, we calculate the real distinct zeros $t_i$ of $T(u, K_1)$ of odd multiplicity for $u \in (-1, 1)$: $-1 < t_1 < \cdots < t_k < +1$.

- Let $i_\delta$, $i_{N_r}$ be the number of zeros of $\delta(z)$ and $N_r(z)$ inside the unit circle, respectively, then we have

\[
i_\delta + i_{N_r} - l = \frac{1}{2} \text{Sgn} [T^{(p)}(-1)] \left( \text{Sgn} [R(-1, K_1, K_2)] \\
+ 2 \sum_{j=1}^{k} (-1)^j \text{Sgn} [R(t_j, K_1, K_2)] + (-1)^{k+1} \text{Sgn} [R(+1, K_1, K_2)] \right).
\]
Stabilization with PD Controllers

- Plant and Controller:  \( P(\hat{z}) = \frac{N(\hat{z})}{D(\hat{z})} \),  \( C(\hat{z}) = \frac{K_1(\hat{z} - K_2)}{\hat{z}} \)

- The characteristic polynomial:  \( \delta(\hat{z}) = \hat{z}D(\hat{z}) + K_1(\hat{z} - K_2)N(\hat{z}) \)

- Consider

\[
\delta(\hat{z})N(\hat{z}^{-1})\big|_{\hat{z}=e^{j\theta}, u=\cos \theta} = R(u, K_1, K_2) + j\sqrt{1-u^2}T(u, K_1)
\]

where

\[
R(u, K_1, K_2) = -uP_1(u) - (1-u^2)P_2(u) - K_1(u + K_2)P_3(u)
\]

\[
T(u, K_1) = K_1P_3(u) + P_1(u) - uP_2(u).
\]

- Parameter separation has again been achieved, that is, \( K_1 \) appears only in the imaginary part and for fixed \( K_1 \) the real part is linear in \( K_2 \).

- Thus the application of the root counting formulas will yield linear inequalities in \( K_2 \), for fixed \( K_1 \).
**STABILIZATION WITH PID CONTROLLERS**

- **PID Controller:**
  \[ C(z) = \frac{K_2 z^2 + K_1 z + K_0}{z(z - 1)} \]

- The characteristic polynomial becomes
  \[ \delta(z) = z(z - 1)D(z) + (K_2 z^2 + K_1 z + K_0) N(z) \]

- Multiplying the characteristic polynomial by \( z^{-1}N(z^{-1}) \),
  \[ z^{-1} \delta(z)N(z^{-1}) = (z - 1)D(z)N(z^{-1}) + (K_2 z + K_1 + K_0 z^{-1}) N(z)N(z^{-1}) \]

- Using the Tchebyshev representations, we have
  \[ z^{-1} \delta(z)N(z^{-1}) = -(u + 1)P_1(u) - (1 - u^2) P_2(u) - [(K_0 + K_2) u - K_1] P_3(u) \]
  \[ + j \sqrt{1 - u^2} \left[ -(u + 1)P_2(u) + P_1(u) + (K_2 - K_0) P_3(u) \right] \]
  \[ = R(u, K_0, K_1, K_2) + j \sqrt{1 - u^2} T(u, K_0, K_2) \]
• Let \( K_3 := K_2 - K_0 \).

• Then \( K_P = -K_1 - 2K_0 \), \( K_I = \frac{K_0+K_1+K_2}{T} \), and \( K_D = K_0T \).

• Hence we rewrite \( R(u, K_0, K_1, K_2) \) and \( T(u, K_0, K_2) \) as follows.

\[
R(u, K_1, K_2, K_3) = -(u + 1)P_1(u) - (1 - u^2) P_2(u) - [(2K_2 - K_3)u - K_1] P_3(u)
\]

\[
T(u, K_3) = P_1(u) - (u + 1)P_2(u) + K_3P_3(u)
\]

• The **parameter separation achieved**: \( K_3 \) appears only in the imaginary part and \( K_1, K_2, K_3 \) appear linearly in the real part.

• Thus by applying root counting formulas to the rational function on the left, and imposing the stability requirement yields **linear** inequalities in the parameters for fixed \( K_3 \).

• The solution is completed by sweeping over the range of \( K_3 \) for which an adequate number of real roots \( t_k \) exist.
Example

- **Plant:** \( G(z) = \frac{1}{z^2 - 0.25} \)

- Then \( R_D(u) = 2u^2 - 1.25, \quad T_D(u) = -2u, \quad R_N(u) = 1, \quad T_N(u) = 0 \)
  \( P_1(u) = 2u^2 - 1.25, \quad P_2(u) = -2u, \quad P_3(u) = 1 \)

- Since \( G(z) \) is of order 2 and \( C(z) \), the PID controller, is of order 2, the number of roots of \( \delta(z) \) inside the unit circle is required to be 4 for stability.

- From Theorem (Root counting for a real polynomial),
  \[ i_i - i_2 = \left( i_\delta + i_{N_r} \right) - (l + 1) \]
  where \( i_\delta \) and \( i_{N_r} \) are the numbers of roots of \( \delta(z) \) and the reverse polynomial of \( N(z) \) inside the unit circle, respectively and \( l \) is the degree of \( N(z) \).
Digital PID Controller Design

- Since the required $i_\delta$ is 4, $i_{N_r} = 0$, and $l = 0$, $i_1 - i_2$ is required to be 3.
- To illustrate the example in detail, we first fix $K_3 = 1.3$.
- Then the real roots of $T(u, K_3)$ in $(-1, 1)$ are $-0.4736$ and $-0.0264$.
- Furthermore, $\text{Sgn}[T(-1)] = 1$, $i_1 - i_2 = 3$ requires that:
  \[ \frac{1}{2} \text{Sgn}[T(-1)] \left( \text{Sgn}[R(-1)] - 2\text{Sgn}[R(-0.4736)] + 2\text{Sgn}[R(-0.0264)] - \text{Sgn}[R(1)] \right) = 3 \]
- We have only one valid sequence satisfying the above equation,
  \[
  \begin{array}{cccc}
    \text{Sgn}[R(-1)] & \text{Sgn}[R(-0.4736)] & \text{Sgn}[R(-0.0264)] & \text{Sgn}[R(1)] \\
    1 & -1 & 1 & -1 \\
  \end{array}
  \]
  $2(i_1 - i_2) = 6$
- From this valid sequence, we have the following set of linear inequalities.
  \[
  \begin{align*}
    -1.3 + K_1 + 2K_2 &> 0 \\
    -0.9286 + K_1 + 0.9472 &< 0 \\
    1.1286 + K_1 + 0.0528K_2 &> 0 \\
    -0.2 + K_1 - 2K_2 &< 0.
  \end{align*}
  \]
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\[
\begin{bmatrix}
K_P \\
K_I \\
K_D
\end{bmatrix}
= \begin{bmatrix}
-2 & -1 & 0 \\
\frac{1}{T} & \frac{1}{T} & \frac{1}{T} \\
T & 0 & 0
\end{bmatrix}
\begin{bmatrix}
K_0 \\
K_1 \\
K_2
\end{bmatrix}
= \begin{bmatrix}
-2 & -1 & 0 \\
\frac{1}{T} & \frac{1}{T} & \frac{1}{T} \\
T & 0 & 0
\end{bmatrix}
\begin{bmatrix}
0 & 1 & -1 \\
0 & 1 & 0 \\
0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
K_1 \\
K_2 \\
K_3
\end{bmatrix}
\]

Stability regions in \((K_1, K_2, K_3)\) space (left) and \((K_P, K_I, K_D)\) space (right)
Maximally Deadbeat Control

- The design scheme attempts to place the closed loop poles in a circle of minimum radius $\rho$. Let $S_\rho$ denote the set of PID controllers achieving such a closed loop root cluster.

- We show below how $S_\rho$ can be computed for fixed $\rho$. The minimum value of $\rho$ can be found by determining the value $\rho^*$ for which $S_{\rho^*} = \emptyset$ but $S_\rho \neq \emptyset, \rho > \rho^*$.

- PID Controller: $\ C(z) = \frac{K_2 z^2 + K_1 z + K_0}{z(z - 1)}$

- The characteristic equation

$$\delta(z) = z(z - 1)D(z) + (K_2 z^2 + K_1 z + K_0) N(z).$$

- Note that

$$D(z)|_{z = -\rho u + j\rho \sqrt{1 - u^2}} = R_D(u, \rho) + j\sqrt{1 - u^2} T_D(u, \rho)$$

$$N(z)|_{z = -\rho u + j\rho \sqrt{1 - u^2}} = R_N(u, \rho) + j\sqrt{1 - u^2} T_N(u, \rho)$$
Digital PID Controller Design

\[ N \left( \rho^2 z^{-1} \right) \bigg|_{z=-\rho u+j\rho \sqrt{1-u^2}} = N(z) \bigg|_{z=-\rho u-j\rho \sqrt{1-u^2}} = R_N(u, \rho) - j \sqrt{1-u^2} T_N(u, \rho). \]

- We evaluate

\[ \rho^2 z^{-1} \delta(z) N \left( \rho^2 z^{-1} \right) = \rho^2 z^{-1} \left[ z(z-1) D(z) + (K_2 z^2 + K_1 z + K_0) N(z) \right] N \left( \rho^2 z^{-1} \right) \]

over the circle \( C_\rho \)

\[ \rho^2 z^{-1} \delta(z) N \left( \rho^2 z^{-1} \right) \bigg|_{z=-\rho u+j\rho \sqrt{1-u^2}} = -\rho^2 (\rho u + 1) P_1(u, \rho) - \rho^3 (1-u^2) P_2(u, \rho) - \left[ (K_0 + K_2 \rho^2) \rho u - K_1 \rho^2 \right] P_3(u, \rho) + j \sqrt{1-u^2} \left[ \rho^3 P_1(u, \rho) - \rho^2 (\rho u + 1) P_2(u, \rho) + (K_2 \rho^2 - K_0) \rho P_3(u, \rho) \right] \]

where

\[ P_1(u, \rho) = R_D(u, \rho) R_N(u, \rho) + (1-u^2) T_D(u, \rho) T_N(u, \rho) \]
\[ P_2(u, \rho) = R_N(u, \rho) T_D(u, \rho) - T_N(u, \rho) R_D(u, \rho) \]
\[ P_3(u, \rho) = R_N^2(u, \rho) + (1-u^2) T_N^2(u, \rho). \]
• By letting \( K_3 := K_2 \rho^2 - K_0 \),

• we have

\[
\rho^2 z^{-1} \delta(z) N (\rho^2 z^{-1}) \bigg|_{z = -\rho u + j \rho \sqrt{1 - u^2}} = -\rho^2 (\rho u + 1) P_1(u, \rho) - \rho^3 (1 - u^2) P_2(u, \rho) - \left[ (2K_2 \rho^2 - K_3) \rho u - K_1 \rho^2 \right] P_3(u, \rho) + j \sqrt{1 - u^2} \left[ \rho^3 P_1(u, \rho) - \rho^2 (\rho u + 1) P_2(u, \rho) + K_3 \rho P_3(u, \rho) \right].
\]

• Fix \( K_3 \), use the root counting formulas, develop linear inequalities in \( K_2, K_3 \) and sweep over the requisite range of \( K_3 \). This procedure is then performed as \( \rho \) decreases until the set of stabilizing PID parameters just disappears.
Example

- We consider the same plant used in the previous example.
- Left figure shows the stabilizing set in the PID gain space at $\rho = 0.275$. 
Digital PID Controller Design

- For a smaller value of \( \rho \), the stabilizing region in PID parameter space disappears. This means that there is no PID controller available to push all closed loop poles inside a circle of radius smaller than 0.275.

- From this we select a point inside the region that is

\[
K_0 = 0.0048, \quad K_1 = -0.3195, \quad K_2 = 0.6390, \quad K_3 = 0.0435.
\]

- From the relationship between parameters, we have

\[
\begin{bmatrix}
    K_P \\
    K_I \\
    K_D
\end{bmatrix}
= \begin{bmatrix}
    -1 & -2\rho^2 & 2 \\
    \frac{1}{T} & \frac{\rho^2}{T} + \frac{1}{T} & -\frac{1}{T} \\
    0 & \frac{\rho^2 T}{T} & -\frac{T}{T}
\end{bmatrix}
\begin{bmatrix}
    K1 \\
    K2 \\
    K3
\end{bmatrix}
= \begin{bmatrix}
    0.3099 \\
    0.3243 \\
    0.0048
\end{bmatrix}
\]

- Right figure shows the closed loop poles that lie inside the circle of radius \( \rho = 0.275 \). The roots are:

\[
0.2500 \pm j0.1118 \quad \text{and} \quad 0.2500 \pm j0.0387.
\]
- We select several sets of stabilizing PID parameters from the set obtained in the previous example (i.e., $\rho = 1$) and compare the step responses between them.
Maximum Delay Tolerance Design

- Finding the maximum values of $L^*$ such that the stabilizing PID gain set that simultaneously stabilizes the set of plants

$$z^{-L}G(z) = \frac{N(z)}{z^L D(z)}, \quad \text{for } L = 0, 1, \cdots, L^*$$

- Let $S_i$ be the set of PID gains that stabilizes the plant $z^{-i}G(z)$. Then $\bigcap_{i=0}^{L} S_i$ stabilizes $z^i G(z)$ for all $i = 0, 1, \cdots, L$.
The right figure shows the stabilizing PID gains when $L = 0, 1$. As seen in the figure, the size of the set is reduced as the delay increases.
Digital PID Controller Design

- In many systems, the set disappears for a large value of $L^*$. This is the maximum delay that can be stabilized by any PID controllers.